A Combinatorial Proof of the Syracuse Conjecture Using Transition Lists

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Abstract

The Syracuse (Collatz) conjecture is a simple yet notoriously difficult open problem in mathematics. Given a positive integer v_0, the associated sequence evolves by the rule: if v_n is even, then $v_{n+1} = v_n/2$ (type 0 transition); otherwise, $v_{n+1} = (3v_n + 1)/2$ (type 1 transition). The conjecture asserts that every such sequence eventually reaches the value 1.

If we consider the set of $2^{f(N)}$ random *transition lists* of length N, each encoding a sequence of type 0 and type 1 transitions in the Syracuse process, then the number of initial values $v_0 < 2^n$ that appear as minimal values generating one of these lists follows the binomial distribution $\mathcal{B}(2^{f(N)}, 2^{N-n})$.

This allows us to bound, using the Central Limit Theorem or the Berry–Esseen inequality, the number of initial values below 2^n that appear as minimal initial values associated with transition lists in the sample.

We introduce the concept of the set $\mathrm{Up}(N, v_0)$, consisting of transition lists for which v_0 is minimal among the first N elements. By studying this set and using the fact that growth remains naturally bounded beyond the first $p > 100$ terms, we show that the cardinality of admissible transition lists is provably insufficient to contradict the conjecture.

Our method is purely discrete and combinatorial, deliberately avoiding classical analytic or ergodic techniques.

1 Introduction

The Syracuse conjecture—also known as the Collatz conjecture or the $3x + 1$ problem—is one of the most well-known unsolved problems in mathematics. Its formulation is deceptively simple: starting from any positive integer v_0, one defines a sequence by the recursive rule

$$v_{n+1} = \begin{cases} v_n/2, & \text{if } v_n \text{ is even (type 0 transition);} \\ (3v_n + 1)/2, & \text{if } v_n \text{ is odd (type 1 transition).} \end{cases}$$

The conjecture states that every such sequence eventually reaches the value 1. Despite its apparent simplicity and extensive computational verification, a general proof has remained elusive.

In this paper, we introduce a rigorous combinatorial model of the Syracuse conjecture, grounded in basic statistical principles. The approach is based on analyzing how initial values v_0 are distributed within large sets of transition lists.

Section 2 introduces the classical Syracuse sequence, its reduced form, and an approximate variant v'_n in which the constant $+1$ is omitted in the odd case. We also define the *transition list* $\mathcal{L}(N, m, d)$—encoding the parity transitions occurring in the Syracuse sequence— and establish a partial order over such lists.

In Section 3, we prove that the number of initial values $v_0 < 2^n$ that appear as minimal values generating one of the $2^{f(N)}$ random *transition lists* of length N—each encoding a sequence of type 0 and type 1 transitions in the Syracuse process—follows the binomial distribution $\mathcal{B}(2^{f(N)}, 2^{N-n})$.

In Section 4, we present the central result—the *Random List Theorem*—which allows us to bound the number of initial values below 2^n that occur as minimal values generating a given transition list in the sample. Two classical methods are employed to estimate the number of minimal initial values: on the one hand, the Central Limit Theorem, which is effective with moderate sample sizes; on the other hand, the Berry–Esseen inequality, which requires significantly larger sets but has the important advantage of enabling a fully formalized proof within a proof assistant.

Section 5 develops a detailed analysis of the approximate sequence v'_n, quantifying the error term $r_n = v_n - v'_n$ and showing that the influence of the omitted constant becomes negligible for large v_0 and moderate N.

The following four sections form the core of the proof:

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- Section 6 studies the boundary list $\mathrm{Ceil}(N)$, which provides a minimal condition ensuring that the approximate sequence v'_n is strictly increasing.
- Section 7 focuses on the minimal list $\mathrm{JGL}(N, v_0)$ in the partial order, guaranteeing growth for the actual sequence v_n. Under suitable conditions, this list coincides with $\mathrm{Ceil}(N)$.
- Section 8 introduces the filtered list $\mathrm{JGL}_{2p}(N, v_0)$, which ensures that the first $p > 100$ values of the sequence satisfy $v_n > v_0/2^{2p}$, a key constraint for bounding type 1 transitions.
- Finally, Section 9 synthesizes these tools to complete the proof. By counting the number of lists above JGL_{2p} and applying the *Random List Theorem*, we derive a contradiction, thereby completing the proof of the conjecture.

In summary, this work offers not only a complete combinatorial proof of the Syracuse conjecture, but also a general methodological framework for approaching related deterministic problems via discrete and statistical techniques.

Beyond the specific case of the $3n + 1$ rule, our method extends naturally to variants such as $3n + b$, divergent rules like $5n + 1$, and other recent conjectures (see arXiv:2107.11160 [5]). These extensions are explored in additional documents (see [1],[2]).

The method developed here is deliberately elementary: it avoids ergodic theory, analytic tools, and arithmetic geometry. Instead, it demonstrates how a purely combinatorial and discrete perspective can lead to a complete and rigorous proof. While it does not aim to rival deep analytic results—such as those of Terence Tao [9]—it highlights the power of structural reasoning in tackling complex dynamical problems. This approach complements analytical techniques and opens new directions for combinatorial insight into the Syracuse problem.

An interactive platform allowing exploration of the numerical properties discussed in this paper is available online (see [3]), although it plays no role in the proofs themselves.

2 Definitions

This section introduces the key definitions that will be used throughout the paper. It provides a reference framework for the construction and proof of the main result.

2.1 Standard Syracuse Sequence: (u_n)

The standard Syracuse sequence (u_n) is defined for any initial value $u_0 > 0$ by the recurrence:

$$u_{n+1} = \begin{cases} \dfrac{u_n}{2}, & \text{if } u_n \text{ is even (type 0 transition)},\\ 3u_n + 1, & \text{if } u_n \text{ is odd (type 1 transition)}. \end{cases}$$

Remarque 2.1. The type of each transition depends on the parity of u_n, which is given by its least significant bit (bit 0).

2.2 Reduced Syracuse Sequence: (v_n)

Since any odd value u_n is followed by an even u_{n+1}, it is natural to compose two consecutive steps into one. This leads to the definition of a reduced sequence (v_n), which simplifies the analysis:

$$v_{n+1} = \begin{cases} \dfrac{v_n}{2}, & \text{if } v_n \text{ is even (type 0)},\\ \dfrac{3v_n + 1}{2}, & \text{if } v_n \text{ is odd (type 1)}, \end{cases} \qquad \text{with } v_0 > 0.$$

Remarque 2.2. The parity of v_n still determines the type of transition. While the sequence could be written as $v_n = T^{(n)}(v_0)$, we retain the recurrence form for clarity.

2.3 Transition List $\mathcal{L}(N, m, d)$

A *transition list* of length N is a sequence of N transition types $t_i \in \{0, 1\}$, representing type 0 and type 1 transitions, respectively. It is denoted:

$$\mathcal{L}(N, m, d) = (t_0, t_1, \ldots, t_{N-1}),$$

where m is the number of type 1 transitions, and $d = N - m$ is the number of type 0 transitions.

- m: total number of type 1 transitions (*multiplications*);
- d: number of type 0 transitions (*divisions by 2*);
- $N = m + d$: total length of the transition list.

For each prefix of the list of length $n \leq N$, we define:

- $m_n = \sum_{i=0}^{n-1} \mathbb{1}_{\{t_i=1\}}$, the number of type 1 transitions among the first n elements;
- $d_n = n - m_n$, the number of type 0 transitions among the first n elements.

Exemple 2.3. For $v_0 = 7$, the sequence is:

$$7 \xrightarrow{1} 11 \xrightarrow{1} 17 \xrightarrow{1} 26 \xrightarrow{0} 13.$$

Then: $\mathcal{L}(4, 3, 1) = (1, 1, 1, 0)$.

Remarque 2.4. The list $\mathcal{L}(N, m, d)$ is also called a *parity vector*, since each t_i corresponds to the least significant bit of v_i.

2.4 Partial Order on Transition Lists

We define a partial order $\preccurlyeq$ on transition lists of length N by comparing the cumulative number of type 1 transitions at each prefix of the list.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two transition lists of length N. We write:

$$\mathcal{L}_1 \preccurlyeq \mathcal{L}_2 \quad \text{if and only if} \quad \text{for all } 0 \leq n \leq N, \quad m_{n,\mathcal{L}_1} \leq m_{n,\mathcal{L}_2},$$

where $m_{n,\mathcal{L}}$ denotes the number of type 1 transitions among the first n elements of list $\mathcal{L}$.

This relation is a partial order: it satisfies reflexivity, antisymmetry, and transitivity.

We also define the associated strict order:

$$\mathcal{L}_1 \prec \mathcal{L}_2 \quad \text{if and only if} \quad \text{for all } 0 \leq n \leq N, \quad m_{n,\mathcal{L}_1} < m_{n,\mathcal{L}_2}.$$

Remarque 2.5. This is not a total order. There may exist two lists $\mathcal{L}_1$ and $\mathcal{L}_2$ such that neither $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ nor $\mathcal{L}_2 \preccurlyeq \mathcal{L}_1$ holds. In such cases, the lists are said to be incomparable under this relation. This situation arises when the distribution of type 1 transitions differs in position but not in number.

Exemple 2.6. Let $\mathcal{L}_1 = (1, 0, 1)$ and $\mathcal{L}_2 = (0, 1, 1)$. The cumulative sums of type 1 transitions yield:

$$(m_1, m_2, m_3) = (1, 1, 2) \quad \text{for } \mathcal{L}_1, \quad \text{and} \quad (0, 1, 2) \quad \text{for } \mathcal{L}_2.$$

Thus, neither $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ nor $\mathcal{L}_2 \preccurlyeq \mathcal{L}_1$ holds: the lists are incomparable.

Remarque 2.7 (Interpretation)**.** This order reflects the temporal positioning of type 1 transitions: a list that accumulates multiplications more slowly (i.e., later in the sequence) is considered "smaller" in this ordering.

2.5 Solutions of a Transition List

We say that the initial or starting value v_0

- *follows* the transition list $\mathcal{L}_N$,
- *realizes* the transition list $\mathcal{L}_N$,
- or is a *solution* of the transition list $\mathcal{L}_N$,

if and only if the first N transitions of the reduced Syracuse sequence starting from v_0 are exactly those specified by $\mathcal{L}_N$.

We say that v_0 is the *minimal solution* of $\mathcal{L}_N$ if $v_0 < 2^N$. The existence of such a solution will be established in Section 3.4.

2.6 Approximate Reduced Syracuse Sequence: (v'_n)

We now introduce an approximate version of the reduced Syracuse sequence by neglecting the constant term in the type 1 transition. Specifically, in place of the expression $3v_n + 1$, we consider only $3v_n$. The resulting sequence (v'_n) is defined by the recurrence:

$$v'_{n+1} = \begin{cases} \dfrac{v'_n}{2}, & \text{if } v_n \text{ is even (type 0)}, \\ \dfrac{3v'_n}{2}, & \text{if } v_n \text{ is odd (type 1)}, \end{cases} \qquad \text{with } v'_0 = v_0.$$

Remarque 2.8. This approximation is especially meaningful when the initial value v_0 is large and the number of steps n remains moderate. Crucially, the transition types of the approximate sequence v' coincide exactly with those of the original sequence v, since the parity (and thus the transition vector (t_i)) is preserved.

However, the values of v'_n may be non-integer, which introduces a discrepancy compared to the actual sequence. To quantify this difference, we define a correction term r_n such that:

$$v_n = v'_n + r_n.$$

This decomposition will be used later to precisely analyze the divergence between the exact and approximate sequences.

3 Binomial Distribution of Initial Values Below a Threshold

Théorème 3.1 (Binomial Distribution of Minimal Initial Solutions)**.** *Let $nb \in \mathbb{N}$, and consider a set of nb independent and distinct transition lists $\mathcal{L}_1, \ldots, \mathcal{L}_{nb}$, each of length N. Assume each list $\mathcal{L}(N, m, d)$ is random, with a proportion $p_{\mathcal{L}} = m/N$ of type 1 transitions. Let $k = N - n$ and R_k denote the number of minimal initial solutions $v_0 < 2^n = 2^{N-k}$ associated with these nb lists.*

Then, for any $0 < k < N - 10$, the random variable R_k follows the binomial distribution:

$$R_k \sim \mathrm{Bin}(nb, 1/2^k).$$

The proof of the theorem is broken down into several intermediate results, presented as lemmas in the following subsections.

3.1 Lemma: the Probability that v_1 is even is $\frac{1}{2}$ for $v_0 \geq 4$

Lemme 3.2. *Let $v_0 \geq 4$ be an integer chosen uniformly in the interval $[2^n, 2^{n+1})$ with $n \geq 2$. Then the parity of v_1, defined by the reduced Syracuse iteration*

$$v_1 = \begin{cases} v_0/2 & \text{if } v_0 \equiv 0 \mod 2, \\ (3v_0 + 1)/2 & \text{if } v_0 \equiv 1 \mod 2, \end{cases}$$

the parity of v_1 is uniformly distributed :

$$\mathbb{P}(v_1 \equiv 0 \mod 2) = \mathbb{P}(v_1 \equiv 1 \mod 2) = \frac{1}{2}.$$

Proof. Let us write the binary decomposition of v_0:

$$v_0 = \sum_{p=0}^{N} a_p \cdot 2^p, \quad \text{with } a_p \in \{0, 1\}.$$

Case 1: v_0 is even ($a_0 = 0$)
Then

$$v_1 = \frac{v_0}{2} = \sum_{p=1}^{N} a_p \cdot 2^{p-1} = \sum_{p=0}^{N-1} a_{p+1} \cdot 2^p.$$

The parity of v_1 is given by a_1. Since $N \geq 4$, the bit a_1 exists and is uniformly distributed in $\{0, 1\}$:

$$\mathbb{P}(v_1 \text{ even } \mid v_0 \text{ even}) = \mathbb{P}(a_1 = 0) = \frac{1}{2}.$$

Case 2: v_0 is odd ($a_0 = 1$)
We have:

$$v_1 = \frac{3v_0 + 1}{2} = \frac{1 + v_0 + 2v_0}{2}.$$

Replacing v_0 by its binary expansion:

$$v_1 = \frac{1 + \sum_{p=0}^{N} a_p \cdot 2^p + \sum_{p=0}^{N} a_p \cdot 2^{p+1}}{2} = \sum_{p=0}^{N+1} a'_p \cdot 2^p.$$

The least significant bit a'_0 depends on:

$$a'_0 = (1 + a_0 + a_1) \mod 2 = (1 + 1 + a_1) \mod 2 = a_1.$$

As in the even case, the parity of v_1 is determined by a_1, which is uniformly random. Hence:

$$\mathbb{P}(v_1 \text{ even } \mid v_0 \text{ odd}) = \mathbb{P}(a_1 = 0) = \frac{1}{2}.$$

□

Remarque 3.3. This lemma shows that the parity of v_1 is exactly balanced as soon as $v_0 \geq 4$, i.e., when the binary representation of v_0 has at least two digits. This is an exact property, not an asymptotic estimate.

Some sources incorrectly state that this equiprobability only holds "in sufficiently large intervals." For instance, the May 2025 version of the French Wikipedia article on the Syracuse conjecture claims:

> *"the parity of the result is independent of that of v, if v is randomly chosen in a sufficiently large interval."*

However, as the above proof shows, the property already holds perfectly for all $v_0 \geq 4$, without any asymptotic assumption.

It is also important to note that this equiprobability cannot be extended to subsequent values v_n, since the trajectory is deterministically correlated with v_0. Assuming independence along the entire sequence is a common error in probabilistic models of the Syracuse dynamics. While the lemma justifies local randomness at the first step, caution is required when extending this reasoning to full orbits.

3.2 Lemma: Bijection between Transition Lists of Length N and Minimal Initial Values $v_0 < 2^N$ That Realize Them

Lemme 3.4. *For every integer $N \geq 1$, there is a bijection between:*

- *the set $\mathcal{L}_N$ of binary transition lists $(t_0, \ldots, t_{N-1}) \in \{0,1\}^N$;*
- *and the set of initial values $v_0 < 2^N$ so that the sequence $(v_1, \ldots, v_N)$ generated by the reduced Syracuse iteration follows the transition pattern $(t_0, \ldots, t_{N-1})$.*

Each transition list uniquely determines a minimal initial value $v_0 < 2^N$ that realizes it. Furthermore, all other values generating the same transition list are of the form:

$$v_0^{(n)} = v_0 + n \cdot 2^N, \quad n \in \mathbb{N}.$$

Remarque 3.5. This relies on extending the definition to include $v_0 = 0$, which is then considered as the minimal solution for all transition lists containing exactly N transitions of type 0 (and no transitions of type 1), instead of assigning $v_0 = 2^N$.

Proof. The reduced Syracuse dynamics assigns to any integer v_0 a transition list $(t_0, \ldots, t_{N-1})$ defined by:

$$t_i = \begin{cases} 0 & \text{if } v_i \text{ is even,} \\ 1 & \text{if } v_i \text{ is odd,} \end{cases}$$

where $v_{i+1} = T(v_i)$ with T the reduced Syracuse function.

We prove by induction on N that for each binary word of length N, there exists a unique minimal $v_0 < 2^N$ realizing it.

Base case $N = 1$ There are two possible transition lists:

- $t_0 = 0$ (even), realized by $v_0 = 0$ (with the extension);
- $t_0 = 1$ (odd), realized by $v_0 = 1$.

Each transition bit is thus realized by a unique $v_0 < 2$.

Inductive step Assume the result holds for lists of length N: for every $\mathcal{L}_N = (t_0, \ldots, t_{N-1})$, there exists a unique minimal value $s_0 < 2^N$ realizing it.

Let $\mathcal{L}_{N+1} = (t_0, \ldots, t_N)$ be a list of length $N + 1$.

By the inductive hypothesis, the prefix $(t_0, \ldots, t_{N-1})$ corresponds to a unique value $s_0 < 2^N$. Consider the two candidate initial values:

$$v_0^{(0)} = s_0, \quad v_0^{(1)} = s_0 + 2^N.$$

Both share the same lower N bits and thus follow the same first N transitions. Let m be the number of type 1 transitions among $(t_0, \ldots, t_{N-1})$. Then, by recurrence[1], their corresponding values at time N differ by 3^m:

$$v_N^{(a)} = s_N + a \cdot 3^m.$$

We now determine which of the two values $v_0^{(a)}$ satisfies t_N, by testing the parity of $v_N^{(a)}$:

- If $s_N \equiv t_N \pmod 2$, choose $a = 0$;
- Otherwise, choose $a = 1$.

Thus, exactly one of the two values $v_0^{(0)}$ or $v_0^{(1)}$ matches the full transition list $\mathcal{L}_{N+1}$, and its value is strictly less than 2^{N+1}.

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[1] Let us detail the first transition:
The value $v_0^{(a)}$ has the same parity as s_0, corresponding to $t_0 \in \{0, 1\}$.

- If $t_0 = 0$, then s_0 is even (since s_0 follows $\mathcal{L}_N$), and

$$v_1^{(a)} = \frac{v_0^{(a)}}{2} = \frac{s_0 + a \cdot 2^N}{2} = \frac{s_0}{2} + a \cdot 2^{N-1} = s_1 + a \cdot 2^{N-1}.$$

- If $t_0 = 1$, then s_0 is odd (since s_0 follows $\mathcal{L}_N$), and

$$v_1^{(a)} = \frac{3v_0^{(a)} + 1}{2} = \frac{3(s_0 + a \cdot 2^N) + 1}{2} = \frac{3s_0 + 1}{2} + \frac{a \cdot 3 \cdot 2^{N-1}}{2} = s_1 + a \cdot 3 \cdot 2^{N-1}.$$

The value $v_1^{(a)}$ has the same parity as s_1, which corresponds to t_1.

One can easily prove by induction that, for all $0 \leq n \leq N$,

$$v_n^{(a)} = s_n + a \cdot 3^{m_n} \cdot 2^{N-n},$$

where m_n denotes the number of type 1 transitions among the first n transitions of $\mathcal{L}_N$.
and for $n = N$:

$$v_N^{(a)} = s_N + a \cdot 3^m.$$

Infinitely many solutions Since adding 2^N does not affect the first N transitions, any integer of the form:

$$v_0^{(n)} = v_0 + n \cdot 2^N, \quad n \in \mathbb{N},$$

also realizes the same transition list. Therefore, for each $\mathcal{L}_N$, there exists an infinite arithmetic progression of initial values with a unique minimal representative in $[0, 2^N)$.

□

Remarque 3.6 (On the precedence of the lemma)**.** In the standard case, this lemma corresponds to results previously established by Riho Terras (1976) [10] and C. J. Everett (1977) [6], as kindly pointed out to me by Shalom Eliahou in a personal correspondence dated December 18, 2024.

These references were not identified in earlier versions of this document (prior to version 3.1.2), as the original articles are written in English and adopt a different formalism.

That said, the main contribution of this section lies in the corollary that follows, which, to the best of our knowledge, constitutes a new result within the specific framework developed here.

3.3 Corollary of Lemma 3.4: $\mathbb{P}(v_0 < 2^{N-1}) = \frac{1}{2}$ for Transition Lists of Length N

Corollaire 3.7. *Let $N \geq 1$. Among all transition lists of length N, the probability that the minimal initial value v_0 satisfies $v_0 < 2^{N-1}$ is exactly*

$$\mathbb{P}(v_0 < 2^{N-1}) = \frac{1}{2}.$$

Proof. We consider only the minimal initial values $v_0 < 2^N$ arising from the bijection of Lemma 3.4.

Given a list of length N, the construction extends a prefix of length $N-1$ by one final bit t_{N-1}. The two candidates for v_0 are:

$$v_0^{(0)} = s_0, \quad v_0^{(1)} = s_0 + 2^{N-1}.$$

Only one of these two values satisfies the final transition, depending on the parity of s_{N-1} and the bit t_{N-1}. The minimal representative $v_0 = s_0$ is selected if and only if:

$$(t_{N-1} = 0 \text{ and } s_{N-1} \text{ is even}) \quad \text{or} \quad (t_{N-1} = 1 \text{ and } s_{N-1} \text{ is odd}).$$

Assuming, as shown in Lemma 3.2, that $\mathbb{P}(s_{N-1} \text{ even}) = \frac{1}{2}$, and letting p denote the probability that $t_{N-1} = 1$, we compute:

$$\mathbb{P}(v_0 = s_0) = (1-p) \cdot \frac{1}{2} + p \cdot \frac{1}{2} = \frac{1}{2}.$$

Hence, among all transition lists of length N, the minimal initial value v_0 falls below 2^{N-1} with probability exactly

$$\mathbb{P}(v_0 < 2^{N-1}) = \frac{1}{2}.$$

□

3.4 Corollary of Lemma 3.4: $\mathbb{P}(v_0 < 2^{N-k}) = \frac{1}{2^k}$ for Transition Lists of Length N

Corollaire 3.8. *Let $N \geq 1$ and $0 \leq k \leq N$. Among all transition lists of length N, the probability that the associated minimal initial value satisfies $v_0 < 2^{N-k}$ is exactly*

$$\mathbb{P}(v_0 < 2^{N-k}) = \frac{1}{2^k}.$$

Proof. By iterating the reasoning of Corollary 3.7 k times, we observe that each additional transition bit splits the space of minimal initial values in half. Starting from the full interval $[0, 2^N)$, the probability that a randomly constructed list yields a minimal v_0 below 2^{N-k} is thus

$$\mathbb{P}(v_0 < 2^{N-k}) = \frac{1}{2^k}.$$

This also yields the following consequences:

- The probability that v_0 falls in the interval $[2^{N-k}, 2^{N-k+1})$ is likewise $\frac{1}{2^k}$;
- By complement, the probability that $v_0 \geq 2^{N-k}$ is $1 - \frac{1}{2^k}$.

Finally, to have a nonzero expected number of minimal values $v_0 < 2^{N-k}$ in a sample of $n_0 = 2^{f(N)}$ transition lists, we require:

$$\frac{2^{f(N)}}{2^k} > 1 \quad \text{if and only if} \quad k < f(N).$$

This inequality gives a critical threshold beyond which the probability of sampling such a value becomes negligible. □

Remarque 3.9. This exact power distribution is crucial in establishing bounds that scale logarithmically with N in the Random List Theorem. It reflects the uniform binary structure induced by the bijection of Lemma 3.4.

3.5 Iterated Binomial Reduction

Lemme 3.10 (Iterated Binomial Reduction)**.** *Let $nb \in \mathbb{N}$, and define a sequence of random variables $(R_k)_{k\geq 0}$ recursively by:*

$$R_0 = nb, \quad \textit{and} \quad R_k \sim \mathrm{Bin}(R_{k-1}, 1/2) \quad \textit{for all } k \geq 1.$$

Then, for every $k \in \mathbb{N}$, the random variable R_k follows the binomial distribution:

$$R_k \sim \mathrm{Bin}\left(nb, \frac{1}{2^k}\right).$$

Proof. We proceed by induction on k.

Base case: for $k = 0$, we have $R_0 = nb$, which is equivalent to $R_0 \sim \mathrm{Bin}(nb, 1)$, i.e., $R_0 \sim \mathrm{Bin}(nb, 1/2^0)$.

Inductive step: suppose that for some $k \geq 0$, we have

$$R_k \sim \mathrm{Bin}\left(nb, \frac{1}{2^k}\right).$$

Then, conditionally on $R_k = r$, the next variable satisfies

$$R_{k+1} \mid R_k = r \sim \mathrm{Bin}(r, 1/2).$$

Thus, we can write:

$$R_{k+1} = \sum_{i=1}^{R_k} Y_i,$$

where the Y_i are independent Bernoulli(1/2) variables, independent of R_k.

Since $R_k \sim \mathrm{Bin}(nb, 1/2^k)$, we can express:

$$R_k = \sum_{i=1}^{nb} X_i, \quad \text{where } X_i \sim \mathrm{Bernoulli}(1/2^k),$$

and the X_i are independent.

Each $X_i = 1$ indicates that the i-th item survived the first k filtering steps. For R_{k+1}, we apply one more independent Bernoulli(1/2) filtering to each $X_i = 1$.

Therefore, each $i \in \{1, \ldots, nb\}$ survives the first $k + 1$ steps with probability:

$$\mathbb{P}(\text{survival}) = \frac{1}{2^k} \cdot \frac{1}{2} = \frac{1}{2^{k+1}}.$$

By independence, we conclude that:

$$R_{k+1} \sim \mathrm{Bin}\left(nb, \frac{1}{2^{k+1}}\right).$$

Conclusion: the result follows by induction: for all $k \in \mathbb{N}$,

$$\boxed{R_k \sim \mathrm{Bin}\left(nb, \frac{1}{2^k}\right).}$$

$\square$

3.6 Proof ot the Theorem

Proof. Each transition list $\mathcal{L}$ defines a unique minimal solution $v_0 < 2^N$ under the convention that $v_0 = 0$ corresponds to the all-zero transition list (see Lemma 3.4).

For each transition t_{N-k-1} in each list, we consider v_0 to be the minimal initial value that solves the first $N - k - 1$ transitions.

We know that $v_0 < 2^{N-k-1}$.

Moreover, v_0 is also the minimal solution for the first $N-k$ transitions of $\mathcal{L}$ if and only if t_{N-k-1} matches the "natural" transition from v_0, that is, if

$$((t_{N-k-1} = 1) \text{ and } v_{N-k-1} \text{ is odd}) \quad \text{or} \quad ((t_{N-k-1} = 0) \text{ and } v_{N-k-1} \text{ is even}).$$

The probability of this event is
$$p_{\mathcal{L}} \cdot \frac{1}{2} + (1 - p_{\mathcal{L}}) \cdot \frac{1}{2} = \frac{1}{2}.$$

Indeed, if the transition does not match, then the minimal solution for the first $N-k$ transitions of $\mathcal{L}$ would be $v_0 + 2^{N-k} \geq 2^{N-k}$, and thus no longer strictly below the threshold.

We now prove by induction that $R_k \sim \text{Bin}\left(nb, \frac{1}{2^k}\right)$.

Base case: For $k = 1$, we consider the final transition t_{N-1} of each transition list. Given that the minimal initial value v_0 for the first $N-1$ transitions satisfies $v_0 < 2^{N-1}$, the value v_0 also solves the full list of N transitions if and only if t_{N-1} matches the natural parity transition induced by v_{N-1}. This occurs with probability $1/2$, since the transition is chosen at random and independently of v_0, and the parity of v_{N-1} is balanced in expectation.

Since the nb transition lists are all distinct and independent, we perform nb independent Bernoulli trials with success probability $1/2$, one for each list. It follows that

$$R_1 \sim \text{Bin}(nb, 1/2).$$

Inductive step: Assume that $R_{k-1} \sim \text{Bin}(nb, 1/2^{k-1})$.

By iterating the same reasoning at step k, after analyzing the last $k-1$ transitions of each list, each remaining minimal value survives the next transition with probability $1/2$, independently. Therefore,

$$R_k \sim \text{Bin}\left(R_{k-1}, \frac{1}{2}\right).$$

Then, by applying Lemma 3.10, we deduce that

$$R_k \sim \text{Bin}\left(nb, \frac{1}{2^k}\right).$$

This completes the proof by induction.

Therefore, we conclude that the number of minimal initial solutions strictly less than 2^{N-k} follows the binomial distribution $\text{Bin}(nb, 1/2^k)$.

□

4 Random List Theorem

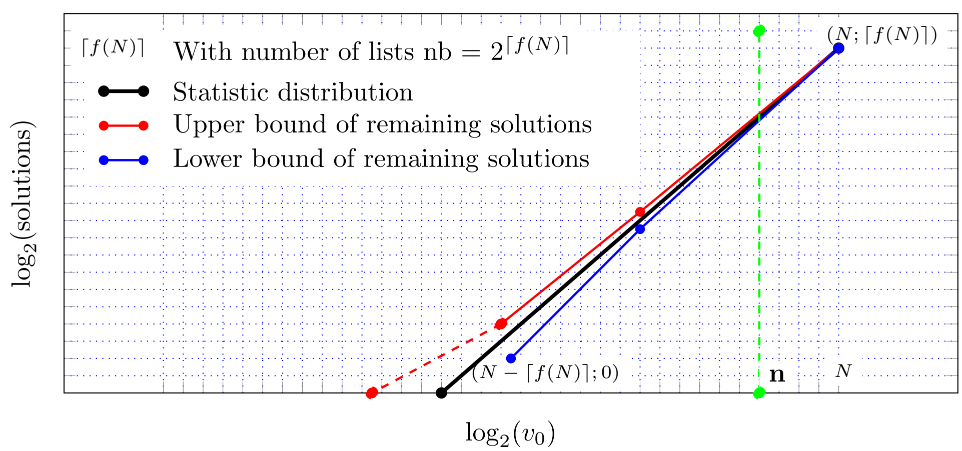

Figure 1: Number of solutions $v_0 < 2^n$

Remarque 4.1 (Idea)**.** The probability that the minimal initial value v_0 of a transition list $\mathcal{L}(N, m, d)$ satisfies $v_0 < 2^n$ is 2^{N-n}.

If $2^{f(N)}$ random lists are tested, then we expect

$$\mathbb{E}\left[\#\{v_0 < 2^n\}\right] \approx 2^{f(N)} \cdot 2^{n-N} = 2^e.$$

Hence, the shift index e provides a direct estimate of the expected number of solutions.

Théorème 4.2 (Random List Theorem)**.** *Let a set of $nb = 2^{f(N)}$ transition lists of length N, independently and randomly generated. Each list $\mathcal{L}(N, m, d)$ may contain an arbitrary proportion m/N of type 1 transitions, without any specific constraint.*

For a given integer $n < N$, let R_n denote the number of minimal initial values $v_0 < 2^n$ among the set of transition lists.

Then R_n follows the binomial distribution:

$$R_n \sim \text{Bin}\left(2^{f(N)}, \frac{1}{2^{N-n}}\right).$$

This distribution follows directly from the independence of the lists and the successive filtering mechanism applied to the last $N - n$ transitions.

Define:

$$e := n - N + \lceil f(N) \rceil.$$

(i) Bounds via the Central Limit Theorem.

Let $4 \le z \le 6$ be a real number. Then, with probability at least $1 - \varepsilon$, where $\varepsilon = e^{-z^2/2}$:

- *if $e \ge 7$, then $\quad R_n \ge 64 - 8\sqrt{2}z$,*
- *if $e \le 6$, then $\quad R_n \le 64 + 8z$.*

(ii) Bounds via the Berry–Esseen inequality.

For any $\varepsilon < 10^{-3}$, define:

$$K := \left\lceil 2 \cdot \log_2\left(\frac{0.56}{\varepsilon}\right)\right\rceil + 1.$$

Then, with probability at least $1 - \varepsilon$, we have:

- *if $e > K$, then $\quad R_n > min := 2^{K-1} - \sqrt{2\ln(1/\varepsilon)} \cdot \sqrt{2^K}$,*
- *if $e < K$, then $\quad R_n < max := 2^K + \sqrt{2\ln(1/\varepsilon)} \cdot \sqrt{2^K}$.*

The following values are guaranteed for some standard thresholds:

ε	K	*min*	*max*
10^{-3}	20	520,481	1,052,383
10^{-4}	26	33,519,272	67,144,024
10^{-5}	33	4,294,522,559	8,590,379,329

Proof. According to Theorem 3.1, we have

$$R_n \sim \text{Bin}(nb, 1/2^{N-n}).$$

(i) Central Limit Theorem approximation:

Let $k = N - n$, the number of suffix transitions under analysis.

We apply the classical Central Limit Theorem to the sum of nb independent and identically distributed Bernoulli variables with constant parameter $p = 1/2^k$.

This sum defines the variable R_n, with expected value and standard deviation given by:

$$\mu := \mathbb{E}[R_n] = nb \cdot p = \frac{nb}{2^k},$$

$$\sigma := \sqrt{\text{Var}(R_n)} = \sqrt{nb \cdot p(1-p)} = \sqrt{\frac{nb}{2^k}\left(1 - \frac{1}{2^k}\right)}.$$

As soon as $\mu = \frac{nb}{2^k} \gtrsim 30$, the normal approximation becomes accurate in practice. Asymptotically, we have convergence in distribution:

$$Z_n := \frac{R_n - \mu}{\sigma} \xrightarrow[nb\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

We now derive probabilistic bounds for R_n using a Gaussian tail threshold $z > 0$.

- **Upper bound (tail on the right):**

$$\mathbb{P}(Z_n < z) > 1 - \varepsilon \quad \text{whenever} \quad R_n < \mu + z \cdot \sigma, \quad \text{with } \varepsilon := 1 - \Phi(z).$$

We bound successively:

$$R_n < \frac{nb}{2^k} + z \cdot \sqrt{\frac{nb}{2^k}\left(1 - \frac{1}{2^k}\right)} < \frac{nb}{2^k} + z \cdot \sqrt{\frac{nb}{2^k}}.$$

Now suppose $nb = 2^{f(N)} \le 2^{\lceil f(N) \rceil}$. Then,

$$R_n < \frac{2^{\lceil f(N) \rceil}}{2^k} + z \cdot \sqrt{\frac{2^{\lceil f(N) \rceil}}{2^k}}.$$

Define $e := n - N + \lceil f(N) \rceil$. Then $e \leq 6$ is equivalent to $k \geq \lceil f(N) \rceil - 6$. Since R_n is decreasing in k, the upper bound is maximal when $k = \lceil f(N) \rceil - 6$. Therefore:

$$\text{if } e \leq 6 \text{ then } R_n < 64 + 8z.$$

– **Lower bound (tail on the left):**
Using the Central Limit Theorem, for any $z > 0$, we have:

$$\mathbb{P}(Z_n > z) > 1 - \varepsilon \quad \text{whenever} \quad R_n > \mu - z \cdot \sigma, \quad \text{with } \varepsilon := 1 - \Phi(z).$$

We start from the inequality:

$$R_n > \frac{nb}{2^k} - z \cdot \sqrt{\frac{nb}{2^k}\left(1 - \frac{1}{2^k}\right)} > \frac{nb}{2^{k-1}} - z \cdot \sqrt{\frac{nb}{2^{k-1}}}.$$

Now suppose $nb = 2^{f(N)} \geq 2^{\lceil f(N) \rceil - 1}$. Then:

$$R_n > \frac{2^{\lceil f(N) \rceil - 1}}{2^k} - z \cdot \sqrt{\frac{2^{\lceil f(N) \rceil}}{2^k}}.$$

Define $e := n - N + \lceil f(N) \rceil$. Then $e \geq 7$ is equivalent to $k \leq \lceil f(N) \rceil - 7$.
Since R_n is decreasing in k, the lower bound is minimal when $k = \lceil f(N) \rceil - 7$. Therefore:

$$\text{if } e \geq 7 \text{ then } R_n > 64 - 8\sqrt{2}z.$$

Numerical remark: For $z \geq 4$, the Mills ratio gives $z \approx \sqrt{2\ln(1/\varepsilon)}$, hence $\varepsilon \approx e^{-z^2/2}$.

(ii) Approximation with Berry–Esseen Inequality:

– **Berry–Esseen Inequality**
We apply the Berry–Esseen inequality to the centered and normalized variable

$$Z_n := \frac{R_n - nb \cdot p}{\sqrt{nb \cdot p(1-p)}},$$

where R_n denotes the number of minimal initial values below 2^n among a large set of $nb = 2^{f(N)}$ transition lists of length N. Although the process is fundamentally deterministic, the distribution of R_n can be approximated by that of a binomial variable $\mathrm{Bin}(nb, p)$, with $p = 1/2^{N-n}$, based on probabilistic modeling of parity transitions.
This allows us to apply the standard form of the Berry–Esseen inequality, which quantifies the convergence to the normal distribution for sums of independent and identically distributed Bernoulli(p) variables.
The third absolute centered moment of a Bernoulli variable is given by

$$\rho = \mathbb{E}[|X - p|^3] = p(1-p)^3 + (1-p)p^3 = p(1-p)(1 - 2p + 2p^2),$$

which is finite for any fixed $p \in (0,1)$. The variance is $\sigma^2 = p(1-p)$, and the Berry–Esseen inequality yields:

$$|\mathbb{P}(Z_n \leq z) - \Phi(z)| \leq \frac{C \cdot \rho}{\sigma^3 \sqrt{nb}} = \frac{C \cdot (1 - 2p + 2p^2)}{(p(1-p))^{1/2} \cdot \sqrt{nb}} = \frac{C_p}{\sqrt{nb}},$$

with $C \leq 0.56$ an absolute constant.
Let

$$C_p := \frac{C \cdot (1 - 2p + 2p^2)}{(p(1-p))^{1/2}},$$

which depends only on p. This formulation enables us to derive explicit quantitative bounds for the probability that R_n deviates from its expectation, using Gaussian approximations with computable error margins.

– **Getting the threshold**
We aim to ensure that $\mathbb{P}(Z_k < z) > 1 - \varepsilon$, and we seek to determine for which values of nb this inequality holds.
Approximating the Gaussian tail for large z using the classical Mills ratio :

$$1 - \Phi(z) \approx \frac{1}{z\sqrt{2\pi}} e^{-z^2/2},$$

we substitute $z := \sqrt{2\ln(1/\varepsilon)}$, which yields:

$$1 - \Phi(z) \approx \frac{\varepsilon}{\sqrt{4\pi \ln(1/\varepsilon)}}.$$

According to the Berry–Esseen inequality:

$$\mathbb{P}(Z_k < z) \geq \Phi(z) - \frac{C_p}{\sqrt{nb}}.$$

Therefore, we require:

$$\Phi(z) - \frac{C_p}{\sqrt{nb}} > 1 - \varepsilon.$$

By substituting the approximation for $\Phi(z)$, we obtain:

$$\frac{\varepsilon}{\sqrt{4\pi \ln(1/\varepsilon)}} + \frac{C_p}{\sqrt{nb}} < \varepsilon.$$

To simplify, note that for small ε, we have $\ln(1/\varepsilon) \gg 1$, so $\frac{\varepsilon}{\sqrt{4\pi \ln(1/\varepsilon)}} \ll \varepsilon$. Therefore, this term becomes negligible, and we may approximate the condition by:

$$\frac{C_p}{\sqrt{nb}} < \varepsilon, \quad \text{which implies} \quad nb > \left(\frac{C_p}{\varepsilon}\right)^2.$$

For large $N-n$ (i.e., when we filter over a large number of final transitions), we have $p = 1/2^{N-n} \ll 1$, and the constant becomes:

$$C_p = \frac{C \cdot (1 - 2p + 2p^2)}{\sqrt{p(1-p)}} \approx \frac{C}{\sqrt{p}}.$$

Substituting this into the bound yields the condition:

$$\boxed{nb > \left(\frac{C}{\varepsilon}\right)^2 \cdot 2^{N-n}.}$$

Taking logarithms (base 2), we obtain:

$$\log_2(nb) > 2\log_2\left(\frac{C}{\varepsilon}\right) + (N - n).$$

Let us define $nb = 2^{f(N)}$. Then the inequality becomes:

$$f(N) > 2\log_2\left(\frac{C}{\varepsilon}\right) + (N - n).$$

This is satisfied as soon as

$$\lceil f(N) \rceil - 1 \geq \left\lceil 2\log_2\left(\frac{C}{\varepsilon}\right) \right\rceil + (N - n).$$

Let us define the threshold:

$$K := \left\lceil 2\log_2\left(\frac{C}{\varepsilon}\right) \right\rceil + 1, \quad \text{and let} \quad e := n - N + \lceil f(N) \rceil.$$

Then the condition becomes simply:

$$e \geq K.$$

– **Upper bound (tail on the right):** By applying the Berry–Esseen inequality at depth $n_K = N - \lceil f(N) \rceil + K$ (i.e., when $e = K$), we obtain:

$$\mathbb{P}(Z_{n_K} \leq z) \geq 1 - \varepsilon, \quad \text{with } z = \sqrt{2\ln(1/\varepsilon)}.$$

Since $Z_{n_K} = \dfrac{R_{n_K} + \mu}{\sigma}$, this implies:

$$R_{n_K} < \mu + z \cdot \sigma, \quad \text{with probability at least } 1 - \varepsilon,$$

where

$$\mu = \mathbb{E}[R_{n_K}] = \frac{nb}{2^{N-n_K}} = \frac{2^{f(N)}}{2^{\lceil f(N) \rceil - K}} \leq 2^K,$$

and

$$\sigma = \sqrt{\mathrm{Var}(R_{n_K})} = \sqrt{\mu\left(1 - \frac{1}{2^{N-n_K}}\right)} < \sqrt{\mu} \leq \sqrt{2^K}.$$

Therefore, with probability at least $1-\varepsilon$, we have:

$$R_{n_K} < 2^K + z \cdot \sqrt{2^K}.$$

Finally, since $R_n \leq R_{n_K}$ for all $e \leq K$ i.e. $n \leq n_K$ (as the sequence R_k is increasing in k), the upper bound on R_{n_K} also applies to R_n.

$$\text{if } e \leq K \text{ then } R_n < 2^K + z \cdot \sqrt{2^K}.$$

– **Lower bound (tail on the left):**
Since

$$|\mathbb{P}(Z_n \leq z) - \Phi(z)| = |\mathbb{P}(Z_n \geq -z) - \Phi(-z)|,$$

we may reuse the previous estimates in the opposite tail.
By applying the Berry–Esseen inequality at depth $n_K = N - \lceil f(N) \rceil + K$ (i.e., when $e = K$), we obtain:

$$\mathbb{P}(Z_{n_K} \geq -z) \geq 1 - \varepsilon, \quad \text{with } z = \sqrt{2\ln(1/\varepsilon)}.$$

Since $Z_{n_K} = \dfrac{R_{n_K} - \mu}{\sigma}$, this implies:

$$R_{n_K} > \mu - z \cdot \sigma, \quad \text{with probability at least } 1 - \varepsilon,$$

where

$$\mu = \mathbb{E}[R_{n_K}] = \frac{nb}{2^{N-n_K}} = \frac{2^{f(N)}}{2^{\lceil f(N) \rceil - K}} \geq 2^{K-1},$$

and

$$\sigma = \sqrt{\mathrm{Var}(R_{n_K})} = \sqrt{\mu\left(1 - \frac{1}{2^{N-n_K}}\right)} < \sqrt{\mu} \leq \sqrt{2^K}.$$

Therefore, with probability at least $1-\varepsilon$, we have:

$$R_{n_K} > 2^{K-1} - z \cdot \sqrt{2^K}.$$

Finally, since $R_n \geq R_{n_K}$ for all $e \geq K$ i.e. $n \geq n_K$ (as the sequence R_k is increasing in k), the lower bound on R_{n_K} also applies to R_n.

$$\text{if } e \geq K \text{ then } R_n > 2^{K-1} - z \cdot \sqrt{2^K}.$$

□

Remarque 4.3 (Random List Theorem for Non-Random Sets of Transition Lists)**.**

Conclusion. For sets of transition lists delimited by suitable boundaries, the *Random List Theorem* can be applied without any special modification.

In the proofs, we would like to apply the *Random List Theorem* to sets of transition lists that are neither random nor independent.

If one were to apply the theorem to the entire set of 2^N transition lists of length N, then for every $0 < n \leq N$ we would obtain $R_n = 2^{N-n}$ by the bijection (see 3.4), and nothing would be random. The difficulty is that if one considers an arbitrary subset of transition lists, without any specific structural property, the extreme cases cannot be excluded, which makes it difficult to draw any meaningful conclusion.

To overcome this difficulty, recall that m_n denotes the number of type 1 transitions among the first n transitions of $\mathcal{L}$. With this notation in place, we shall apply the *Random List Theorem* to a family of transition lists $\mathcal{L}(N, m, d)$ satisfying the condition

$$m_n \geq \lceil kn \rceil \quad \text{for all } 0 < n \leq N,$$

together with either $\lceil kN \rceil \leq m \leq N$ or $m = \lceil kN \rceil$, where $k = \ln(2)/\ln(3)$, for instance for the list Ceil(N) that we shall study later in Section 6.

Each such list, as in Figure 2, can be interpreted as a discrete path from $(0,0)$ to (d,m) consisting of N elementary steps, where each step is either:

- a horizontal move (type 0 transition), increasing d by 1; or
- a vertical move (type 1 transition), increasing m by 1.

In the diagram above:

- The blue line represents the Ceil boundary (a constraint to be respected);

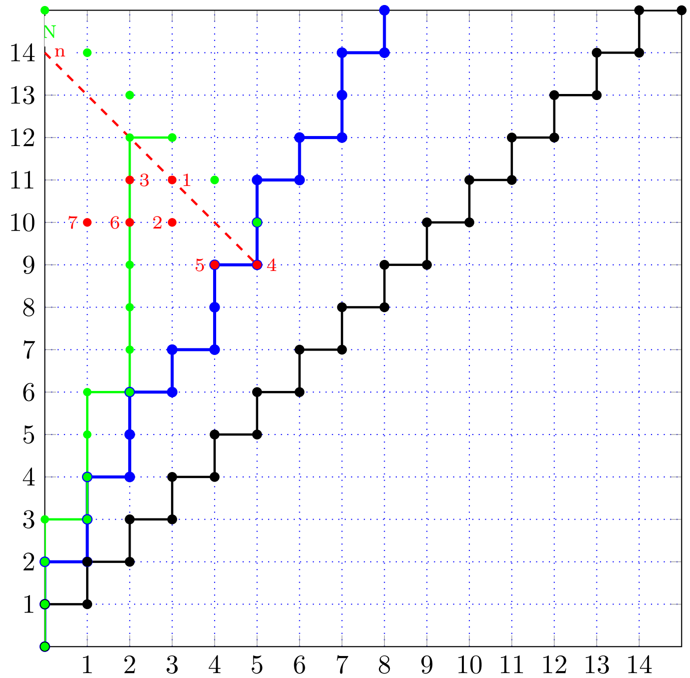

Figure 2: Diagram of transition paths relative to the Ceil boundary.

- The green line is a valid transition list, always staying above Ceil;
- The black line is the classical boundary of the Catalan triangle (without the Ceil constraint);
- Green points indicate the endpoints of valid transition lists for $N = 15$ (only the intersection point with Ceil(N) when we restrict to $m = k \cdot N$);

Note that transition lists passing through the points on the vertical axis $(0, n)$ have the minimal solution $v_0 \geq 2^{n-1}$.

The number of lists passing through each point (d, m) is at least on the order of N, which is very large, except at $(0, m)$ where m is the extremal value of m; in that case, there is only a single list, but its minimal solution is far too large and does not belong to the set of admissible solutions.

For $n = m + d > 2$, at the point (d, m) the probability that v_n is even is equal to $1/2$.

For the minimal solution v_0 of a transition list to be less than 2^{n-1}, and therefore equal to the value v_0 obtained for the restriction to the first $n-1$ transitions, it is necessary that the transition t_{n-1} be the "natural" transition taking v_{n-1} to v_n.

Conversely, for the minimal solution v_0 of a transition list to be greater than 2^{n-1}, and therefore not equal to the value v_0 obtained for the restriction to the first $n-1$ transitions, it suffices that the transition t_{n-1} is not the natural transition from v_{n-1} to v_n.

- For most points (interior points), such as point 1 with coordinates $(3, 11)$: the lists passing through point 1 originate either from point 2 or from point 3.
- For points on the boundary that are preceded by a single "East" step, such as point 4 with coordinates $(5, 9)$: the lists passing through point 4 originate only from point 5.
- In the case where the maximal value of m is taken to be $\lceil kN \rceil$, then for points with maximal ordinate, such as point 6 with coordinates $(2, 10)$: the lists passing through point 6 originate only from point 7.

In all these situations, a very large number of lists pass through the point, which means that the probability of v_{n-1} being even is always $1/2$. For the minimal solution v_0 of a transition list to satisfy $v_0 < 2^{n-1}$, it is necessary that the considered transition be the natural one, i.e., that v_{n-1} is odd at point 2 or even at point 3. Hence, statistically, there are twice as few lists after accounting for transition t_{n-1} whose minimal solution is less than 2^{n-1} as there were with minimal solution less than 2^n before accounting for this transition.

Repeating the same reasoning for all the last transitions, we conclude that for these sets of non-random and non-independent lists, the same result holds as in the random case.

We may note that translating the Ceil boundary horizontally to the right by prefixing it with $2p$ type 0 transitions does not alter the previous argument.

Under these circumstances, the *Random List Theorem* can be applied without any special modification.

Remarque 4.4 (Heuristic Approach to Establishing the Existence of Solutions)**.** Using the Central Limit Theorem, we observe that in the case $e = 6$, which is equivalent to $N - n = \lceil f(N) \rceil - 6$ and hence $2^{\lceil f(N) \rceil}/2^{N-n} = 64$, we obtain:

$$R_n < 64 + 8z = 96 < 128 = 2^7 \quad \text{for } z = 4.$$

This indicates that the number of minimal values is almost halved at each step when analyzing the last $N - n$ transitions. What initially appeared chaotic at the individual level becomes a smooth continuum when considering the system globally.

Even though there is no rigorous mathematical justification for it, the process being deterministic allows us to reasonably conjecture that, by adding 7+6=13 more steps (to account for the remaining fluctuations), we reach $R_{n-13} = 0$, meaning that there are no solutions $v_0 < 2^{n-13}$.

From this, we heuristically infer the following rule:

$$\boxed{\text{If } e < -7, \text{ then } R_n = 0 \text{ (no solution } v_0 < 2^n) \text{ with very high probability.}}$$

The probability is increasing as $e \ll -7$.

This rule is not mathematically rigorous, but it provides a useful intuition before applying formal reasoning with larger residual solutions.

Remarque 4.5. The validity of this estimate relies on the assumption that the sample of transition lists is drawn uniformly at random. Biases in the selection—such as favoring lists associated with small v_0—can significantly distort the statistical outcome. This has been confirmed by discrepancies observed in numerical experiments based on non-uniform or partitioned samplings.

Remarque 4.6. In earlier versions of this document (up to version 4.2.1 inclusive), the probabilistic reasoning relied on Corollary 3.8, which states that $\mathbb{P}(v_0 < 2^{N-k}) = \frac{1}{2^k}$. To bound the number of values $v_0 < 2^{N-k}$, denoted by R_k, the last k transitions were considered, and the Central Limit Theorem was used to estimate the associated binomial distribution at each step.

At each stage, R_k was bounded above and below around the expected proportion, using an interval centered at $n/2$ with growing width. This allowed a valid interval to be maintained at each step, but without control over the global error probability.

The weakness of this approach lies in the fact that extreme cases (beyond a certain number of standard deviations) were not taken into account. The assumption that R_k could not fall outside this interval relied on the idea that extreme cases could not occur, due to the underlying process being deterministic rather than purely random — a mathematically incorrect reasoning.

Indeed, if one fixes a threshold $z_k = 4$, corresponding to a local error $\varepsilon_k \approx 3.35 \times 10^{-4}$, then the probability that at least one of the k steps falls outside the interval is bounded above by $k\varepsilon_k$ (since the probability of a union is less than the sum of individual probabilities). For significant values of k (as used in the proof with $\alpha = 20$, $c_\alpha = 285$, $p = 100$, giving $k = c_\alpha \cdot \alpha - (c_\alpha + p) = 285 \times 20 - (285 + 100) = 5580$), this leads to a global error greater than 1, rendering the argument invalid.

In the current version, this mistake is addressed by consolidating the k steps into a single argument, relying on the fact that $R_k \sim \text{Bin}\left(nb, \frac{1}{2^k}\right)$ (see Theorem 3.1).

Remarque 4.7 (Comparison between the asymptotic (Central Limit Theorem) and rigorous (Berry–Esseen) approaches)**.** In informal reasoning, it is common to apply the Central Limit Theorem (CLT) to approximate a binomial distribution by a normal distribution as soon as the condition

$$nb \cdot p \gtrsim 30$$

is met. In our context, this allows filtering up to $N - n = f - 6$ when $nb = 2^f$, leaving only

$$R_{f-6} \approx 2^6 = 64$$

residual elements to analyze.

However, this approximation relies on asymptotic convergence without any explicit error bound. It is therefore not directly usable in a formal proof system such as Coq or Lean.

In contrast, the Berry–Esseen inequality provides a fully explicit bound on the deviation from the normal distribution. When applied with $\varepsilon = 10^{-3}$, it restricts the filtering depth to

$$N - n = f - 20,$$

leaving a much larger number of residual elements:

$$R_{f-20} \approx 2^{20} \approx 10^6.$$

This loss of efficiency is the price to pay for obtaining a **rigorous and formally justifiable** upper bound on the error probability, which is essential for formal verification.

Summary: the CLT provides sharper bounds but is not formally provable; Berry–Esseen is more conservative but suitable for rigorous proofs.

5 The Approximate Reduced Syracuse Sequence: (v'_n)

We consider an approximate version of the reduced Syracuse sequence, where the term $3v_n + 1$ is replaced with $3v_n$. This approximation is intuitively justified when v_0 is sufficiently large and n remains moderate, in which case the additive term $+1$ becomes negligible compared to the dominant multiplication by 3.

We construct a sequence (v'_n) that reproduces the same transition types (even or odd) as the exact sequence $(v_n)_{n\geq 0}$. It is defined by:

$$\begin{cases} v'_0 = v_0 > 0, \\ v'_{n+1} = \dfrac{v'_n}{2} & \text{if } v_n \text{ is even (type 0 transition),} \\ v'_{n+1} = \dfrac{3v'_n}{2} & \text{if } v_n \text{ is odd (type 1 transition).} \end{cases}$$

Note that the elements of the approximate sequence (v'_n) are generally not integers.

5.1 Decomposition of v_n in Terms of v'_n and a Rational Residue

Proposition 5.1. *For all $n \geq 0$, there exists a rational number $r_n \in \mathbb{Q}$ such that*

$$v_n = v'_n + r_n.$$

Proof. The proposition holds at $n = 0$ with $r_0 = 0$.

Assume it holds for some $n \geq 0$: $v_n = v'_n + r_n$. We prove it holds at $n + 1$:

- **If v_n is even:**
$$v_{n+1} = \frac{v_n}{2} = \frac{v'_n + r_n}{2} = \frac{v'_n}{2} + \frac{r_n}{2} = v'_{n+1} + \frac{r_n}{2}.$$
So $r_{n+1} = \frac{r_n}{2}$.
- **If v_n is odd:**
$$v_{n+1} = \frac{3v_n + 1}{2} = \frac{3(v'_n + r_n) + 1}{2} = \frac{3v'_n}{2} + \frac{3r_n + 1}{2} = v'_{n+1} + \frac{3r_n + 1}{2}.$$
So $r_{n+1} = \frac{3r_n+1}{2}$.

By induction, the proposition holds for all $n \geq 0$. □

Remarque 5.2. The sequence (r_n) can be defined recursively based on the transition types of (v_n):

$$\begin{cases} r_0 = 0 \\ r_{n+1} = \dfrac{r_n}{2} & \text{if } v_n \text{ is even} \\ r_{n+1} = \dfrac{3r_n + 1}{2} & \text{if } v_n \text{ is odd} \end{cases}$$

Note that the recurrence relation for (r_n) depends only on the parity pattern of (v_n) (i.e., the transition list), and not on the actual values of (v'_n) or the initial value v_0. It acts as a rational "residue" that encodes the discrepancy and allows reconstruction of the exact sequence (v_n) from its approximation (v'_n).

In particular, $r_n \geq 0$ for all $n \geq 0$.

Remarque 5.3. The sequence (r_n) remains small compared to (v'_n) when v_0 is large and n is moderate, justifying the approximation $v_n \approx v'_n$. This observation will be quantified in the next section to control the error term in applications of the approximate model.

5.2 Explicit Expression of v_n in Terms of v_0 and r_n

Proposition 5.4. *Let $\mathcal{L}(N, m, d) = (t_0, \ldots, t_{N-1})$ be a transition list of length N, and let m_n denote the number of type 1 transitions among its first n entries. Then for all $0 \leq n \leq N$, we have:*

$$v_n = \frac{3^{m_n}}{2^n} v_0 + r_n,$$

where (r_n) is the sequence defined in Proposition 5.1.

Proof. We recall that v'_n evolves under multiplicative factors of 1/2 and 3/2, depending on the transitions. After m_n type 1 transitions and $(n - m_n)$ type 0 transitions, we have:

$$v'_n = \left(\frac{3}{2}\right)^{m_n} \left(\frac{1}{2}\right)^{n-m_n} v_0 = \frac{3^{m_n}}{2^n} v_0.$$

Using $v_n = v'_n + r_n$, the result follows. □

Remarque 5.5. This decomposition highlights a multiplicative factor $3^m/2^N$ depending only on the global structure of the transition list $\mathcal{L}(N,m,d)$, and a residue r_N depending solely on the positions of the type 1 transitions—not on the initial value v_0.

This is a key step toward applying the *Random List Theorem* discussed in Section 4.

5.3 Closed-Form Expression for r_n Based on Transitions

Théorème 5.6. *Let $\mathcal{L}(N,m,d)$ be a transition list of length N. For any $0 \le n \le N$, let m_n denote the number of type 1 transitions among the first n elements, and let $\mathrm{ind}(i)$ denote the index (starting from 0) of the i^{th} type 1 transition in the list. Then:*

- *If $m_n = 0$, then $r_n = 0$.*
- *If $m_n > 0$, then:*

$$r_n = \frac{3^{m_n}}{2^n} \sum_{i=1}^{m_n} \frac{2^{\mathrm{ind}(i)}}{3^i}.$$

Proof. We proceed by induction on n.

Base case: $n = 1$

- For $\mathcal{L} = (0)$, $m_1 = 0$ and $r_1 = 0$, so the formula holds (empty sum).
- For $\mathcal{L} = (1)$, $m_1 = 1$, $\mathrm{ind}(1) = 0$:

$$r_1 = \frac{3^1}{2^1} \cdot \frac{2^0}{3^1} = \frac{1}{2}.$$

which matches the closed-form expression for r_1.

Induction step: Assume the formula holds at rank n. We show it holds at $n+1$:

- If $t_n = 0$, then $m_{n+1} = m_n$, and:

$$r_{n+1} = \frac{r_n}{2} = \frac{3^{m_n}}{2^{n+1}} \sum_{i=1}^{m_n} \frac{2^{\mathrm{ind}(i)}}{3^i}.$$

- If $t_n = 1$, then $m_{n+1} = m_n + 1$, and:

$$r_{n+1} = \frac{3r_n + 1}{2}.$$

Substituting r_n:

$$\begin{aligned} r_{n+1} &= \frac{1}{2^{n+1}} \left(3 \cdot \sum_{i=1}^{m_n} 3^{m_n - i} \cdot 2^{\mathrm{ind}(i)} + 2^n \right), \\ &= \frac{1}{2^{n+1}} \left(\sum_{i=1}^{m_n} 3^{m_{n+1} - i} \cdot 2^{\mathrm{ind}(i)} + 3^0 \cdot 2^n \right), \\ &= \frac{1}{2^{n+1}} \sum_{i=1}^{m_{n+1}} 3^{m_{n+1} - i} \cdot 2^{\mathrm{ind}(i)}. \end{aligned}$$

Thus, the formula holds at $n+1$. □

5.4 Effect of the Order of Type 0 Transitions on the Growth of r_n

Proposition 5.7. *Among all transition lists $\mathcal{L}(N,m,d)$ with m type 1 and d type 0 transitions, the final residue r_N satisfies:*

- *r_N is minimal when all type 1 transitions occur first (denoted LRmin),*
- *r_N is maximal when all type 0 transitions occur first (denoted LRmax).*

In particular:

$$r_N^{\min} = \frac{3^m}{2^N} - \frac{1}{2^d}, \qquad r_N^{\max} = \left(\frac{3}{2}\right)^m - 1.$$

Proof. From Theorem 5.6, we write:

$$r_N = \frac{3^m}{2^N} \sum_{i=1}^{m} \frac{2^{\mathrm{ind}(i)}}{3^i}.$$

Shifting a type 0 transition earlier increases some indices $\mathrm{ind}(i)$ without decreasing any. Since $x \mapsto 2^x$ is strictly increasing, r_N increases accordingly.

Minimum: all type 1 transitions first:

$$\mathrm{ind}(i) = i - 1, \quad \text{for } 1 \le i \le m.$$

$$r_N^{\min} = \frac{3^m}{2^N}\sum_{i=1}^{m}\frac{2^{i-1}}{3^i} = \frac{3^m}{2^N}\cdot\sum_{i=1}^{m}\left(\frac{2}{3}\right)^{i-1}\cdot\frac{1}{3} = \frac{3^m}{2^N}\cdot\left(\frac{1-\left(\frac{2}{3}\right)^m}{1-\frac{2}{3}}\right)\cdot\frac{1}{3}.$$
$$= \frac{3^m}{2^N} - \frac{1}{2^d}.$$

Maximum: all type 0 transitions first:

$$\mathrm{ind}(i) = d + i - 1.$$

We factor out the 2^d term:

$$r_N^{\max} = \frac{3^m}{2^N}\sum_{i=1}^{m}\frac{2^{d+i-1}}{3^i} = \frac{2^d\cdot 3^m}{2^N}\sum_{i=1}^{m}\frac{2^{i-1}}{3^i}.$$

This sum is the same geometric series as above, hence:

$$r_N^{\max} = \frac{2^d\cdot 3^m}{2^N}\left(1-\left(\frac{2}{3}\right)^m\right) = \left(\frac{3}{2}\right)^m - 1.$$

□

Remarque 5.8. The order of type 0 transitions can exponentially influence the residue r_N. Between the two extreme configurations:

$$\frac{r_N^{\max}}{r_N^{\min}} \approx 2^d.$$

This justifies focusing on subsets of transition lists where the residue r_N remains uniformly bounded. Such control is essential when comparing the exact trajectory (v_n) to its approximation (v'_n).

5.5 Final Residue for a Concatenation of Transition Lists

In this section, we study how the final residue $R_0 = r_{N_0}$ evolves when the transition list $\mathcal{L}_0 = \mathcal{L}(N_0, m_0, d_0)$ is obtained by concatenating a collection of sublists $\mathcal{L}_1, \ldots, \mathcal{L}_n$.

For each $k = 1, \ldots, n$, we define:

- $\mathcal{L}_k = \mathcal{L}(N_k, m_k, d_k)$: a transition list of length $N_k = m_k + d_k$,
- $F_k = \dfrac{3^{m_k}}{2^{N_k}}$: the multiplicative factor associated with $\mathcal{L}_k$,
- $R_k = r_{N_k}$: the rational residue associated with $\mathcal{L}_k$,
- $f_k = \prod_{i=1}^k \dfrac{1}{F_i}$: the reciprocal product of the F_i up to index k.

We recall from Proposition 5.4 that the final value of a block of transitions satisfies:

$$v_{(N_k,\mathcal{L}_k)} = F_k\cdot v_{(0,\mathcal{L}_k)} + R_k.$$

Proposition 5.9 (Concatenation formula for residues)**.** *Let $\mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2 + \cdots + \mathcal{L}_n$ be the successive concatenation of the lists $\mathcal{L}_k$. Then the final residue R_0 associated with $\mathcal{L}_0$ satisfies:*

$$R_0 = F_0\cdot\sum_{k=1}^{n} f_k R_k, \quad \textit{where } F_0 = \prod_{k=1}^{n} F_k = \frac{3^{m_0}}{2^{N_0}}.$$

Proof. We prove the result by induction on the number n of concatenated blocks.

Base case: $n = 2$ Let $\mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2$. From Proposition 5.4, we have:

$$v_{(N_2,\mathcal{L}_2)} = F_2\cdot v_{(0,\mathcal{L}_2)} + R_2, \quad \text{and} \quad v_{(0,\mathcal{L}_2)} = v_{(N_1,\mathcal{L}_1)} = F_1\cdot v_{(0,\mathcal{L}_1)} + R_1.$$

Therefore:

$$\begin{aligned} v_{(N_0,\mathcal{L}_0)} &= F_2(F_1 v_0 + R_1) + R_2 \\ &= F_0 v_0 + F_2 R_1 + R_2, \\ \text{so} \quad R_0 &= F_2 R_1 + R_2 = F_0(f_1 R_1 + f_2 R_2). \end{aligned}$$

The general case follows by iterating this recurrence. □

Corollaire 5.10 (Repeated iterations of a fixed block)**.** *If $\mathcal{L}_0 = n\cdot\mathcal{L}_1$ (concatenation of n identical copies of $\mathcal{L}_1$), then the total residue is given by:*

$$R_0 = R_1\cdot\sum_{k=0}^{n-1} F_1^k = R_1\cdot\frac{1-F_1^n}{1-F_1}.$$

Proof. This is a special case of Proposition 5.9, where $F_k = F_1$ and $R_k = R_1$ for all k. Hence:

$$R_0 = F_1^n \cdot \sum_{k=1}^{n} \frac{R_1}{F_1^k} = R_1 \cdot \sum_{k=0}^{n-1} F_1^k.$$

□

Remarque 5.11. If $F_1 \approx 1$, let $F_1 = 1 - u_1$ with $u_1 \ll 1$. Then:

$$F_1^n \approx 1 - nu_1, \quad \text{so} \quad R_0 \approx nR_1,$$

yielding a linear approximation for the total residue.

Proposition 5.12 (Concatenation with arbitrary multiplicities)**.** *Let*

$$\mathcal{L}_0 = \sum_{k=1}^{n} p_k \cdot \mathcal{L}_k + \mathcal{L}_{n+1}, \quad \textit{with } p_k \in \mathbb{N}^*.$$

Then the total residue is given by:

$$R_0 = F_0 \cdot \left(\sum_{k=1}^{n} \frac{R_k}{\prod_{j=1}^{k} F_j^{p_j}} \cdot \sum_{i=0}^{p_k - 1} F_k^i \right) + R_{n+1}.$$

Proof. Expand each block $p_k \cdot \mathcal{L}_k$ as p_k successive copies and apply Corollary 5.10 to each. The total contribution from block k is scaled by the product of the inverse multiplicative factors from previous blocks.

□

Remarque 5.13. If each $F_k \approx 1$, writing $F_k = 1 - u_k$ with $u_k \ll 1$, we obtain the approximation:

$$R_0 \approx F_0 \cdot \sum_{k=1}^{n} \frac{p_k R_k}{\prod_{j=1}^{k} F_j^{p_j}} + R_{n+1}.$$

In particular, when $\prod_{j=1}^{k} F_j^{p_j} \approx 1$, this simplifies to:

$$R_0 \approx F_0 \cdot \sum_{k=1}^{n} p_k R_k + R_{n+1}.$$

This quasi-linear behavior of the total residue is a useful heuristic in probabilistic models involving repeated motifs.

Remarque 5.14. The concatenation formulas derived in this section provide a powerful tool to compute the residue r_N of complex transition lists by decomposing them into elementary blocks. This modularity will be instrumental in the analysis of filtered or structured transition patterns in subsequent sections.

6 Study of the Transition List $\mathrm{Ceil}(N)$

We focus here on a particular transition list, denoted $\mathrm{Ceil}(N)$, defined by a strict control on the proportion of type 1 transitions.

Définition 6.1. Let m_n be the number of type 1 transitions among the first n transitions in a list $\mathcal{L}(N, m, d)$. The list $\mathrm{Ceil}(N)$ is defined by the condition:

$$m_n = \left\lceil \frac{\ln 2}{\ln 3} \cdot n \right\rceil \quad \text{for all } 0 < n \leq N.$$

6.1 Threshold of the Trajectory: $v_n > v_0$ for All $0 < n \leq N$

Proposition 6.2. *Let (v_n) be the Syracuse sequence associated with the list* $\mathrm{Ceil}(N)$*. Then:*

$$v_n > v_0 \quad \textit{for all } 0 < n \leq N.$$

Proof. By definition of $\mathrm{Ceil}(N)$, we have for all $n > 0$:

$$m_n = \left\lceil \frac{\ln 2}{\ln 3} \cdot n \right\rceil > \frac{\ln 2}{\ln 3} \cdot n.$$

This implies:

$$m_n \cdot \ln 3 > n \cdot \ln 2 \quad \text{if and only if} \quad 3^{m_n} > 2^n.$$

Using Proposition 5.4, we write:

$$v_n = \frac{3^{m_n}}{2^n} \cdot v_0 + r_n,$$

with $r_n \geq 0$. Therefore:

$$v_n > v_0,$$

as required. □

6.2 Characterization of Type 0 Transitions in $\mathrm{Ceil}(N)$

Proposition 6.3. *In the list* $\mathrm{Ceil}(N)$*, the transition* t_n *is of type 0 if and only if the factor*

$$F_n = \frac{3^{m_n}}{2^n}$$

satisfies $F_n \geq 2$*, for all* $0 < n \leq N$*.*

Proof. We analyze the condition $F_n \geq 2$:

$$\frac{3^{m_n}}{2^n} \geq 2 \quad \text{if and only if} \quad \frac{3^{m_n}}{2^{n+1}} \geq 1 \quad \text{if and only if} \quad m_n \cdot \ln 3 \geq (n+1) \cdot \ln 2.$$

This is equivalent to:

$$m_n \geq \frac{\ln 2}{\ln 3} \cdot (n+1) \quad \text{if and only if} \quad m_n \geq \left\lceil \frac{\ln 2}{\ln 3} \cdot (n+1) \right\rceil = m_{n+1}.$$

Since the list $\mathrm{Ceil}(N)$ satisfies $m_n \leq m_{n+1}$ by construction, equality must hold: $m_n = m_{n+1}$. Therefore, t_n is a type 0 transition.

Conversely, if t_n is of type 0, then $m_n = m_{n+1}$, which implies:

$$m_n \geq \left\lceil \frac{\ln 2}{\ln 3} \cdot (n+1) \right\rceil,$$

and thus:

$$\frac{3^{m_n}}{2^n} \geq 2.$$

Hence, the equivalence is proved. □

6.3 Block Decomposition of $\mathrm{Ceil}(n)$

Lemme 6.4. *Let the following constants be given:*

- $\mathrm{N} = 1054$*,*
- $a = 484$*, and* $p < \mathrm{N}$*,*
- $\mathcal{L}_1 = \mathrm{Ceil(N)}$*,*
- $\mathcal{L}_2 = \mathrm{Ceil}(a)$ *followed by a type 0 transition,*
- $\mathcal{L}_3 = \mathrm{Ceil}(p)$*.*

Define, for all $1 \leq k \leq n$*:*

$$q_k = \left\lceil \frac{k(\ln 2 - \ln F_a) - \sum_{i=1}^{k-1} q_i \ln F_{\mathrm{N}}}{\ln F_{\mathrm{N}}} \right\rceil,$$

where $F_n = \frac{3^{m_n}}{2^n}$ *with* $m_n = \left\lceil \frac{\ln 2}{\ln 3} \cdot n \right\rceil$ *in the list* $\mathrm{Ceil}(n)$*.*

Then, for all $n \geq 1$*:*

$$\mathrm{Ceil}\left(\sum_{k=1}^{n} q_k \mathrm{N} + n(a+1) + p \right) = \sum_{k=1}^{n} (q_k \mathcal{L}_1 + \mathcal{L}_2) + \mathcal{L}_3.$$

Proof. The proof is by induction on n.

Case $n = 1$ (with $p = 0$): We seek the smallest value of q_1 such that one of the transitions in the repeated pattern Ceil(N) reaches the mutation threshold, i.e., becomes a type 0 transition when $F_n \geq 2$.

We consider $F_{q\mathrm{N}+n} = (F_{\mathrm{N}})^q \cdot F_n$, where $F_{\mathrm{N}} = F_{1054} \approx 1.00004$. Among all values $F_n = \frac{3^{m_n}}{2^n}$ for $0 \leq n < \mathrm{N}$ that are strictly less than 2, the three largest are:

$n+1$	F_n	$\ln F_n$	$\frac{\ln 2 - \ln F_n}{\ln F_{\mathrm{N}}}$
485	1.99795657	0.69212494	23.4168
401	1.99378892	0.69003681	71.2504
317	1.98962997	0.68794868	119.0839

We observe that $F_{484} \approx 1.99796$ is the largest value strictly below 2, so we set $a = 484$. Any earlier transition t_b (with $b < a$) satisfies $F_b < F_a$ and therefore reaches the mutation threshold later.

We now determine the smallest integer q_1 such that:

$$F_{q_1 \mathrm{N}+a} = (F_\mathrm{N})^{q_1} \cdot F_a \geq 2,$$

which leads to:

$$q_1 \geq \frac{\ln 2 - \ln F_a}{\ln F_\mathrm{N}} \quad \text{so} \quad q_1 = \left\lceil \frac{\ln 2 - \ln F_a}{\ln F_\mathrm{N}} \right\rceil.$$

Using the numerical values:

$$\begin{aligned} \ln F_a &\approx 0.69212494, \\ \ln F_\mathrm{N} &\approx 0.00004365, \\ \text{so} \quad q_1 &= \left\lceil \frac{0.693147 - 0.69212494}{0.00004365} \right\rceil = 24. \end{aligned}$$

It is important to justify why it is always the transition at position a that mutates first. This follows from:

- F_a being the largest among the $F_n < 2$, so $(F_\mathrm{N})^q \cdot F_a$ exceeds 2 before any other $(F_\mathrm{N})^q \cdot F_b$;
- and the construction of $\mathrm{Ceil}(n)$, which ensures that when multiple transitions simultaneously cross the threshold, the leftmost one is selected.

Thus, transition t_a mutates first, and we have:

$$\mathrm{Ceil}(q_1\mathrm{N} + a + 1) = q_1 \cdot \mathrm{Ceil}(\mathrm{N}) + \mathrm{Ceil}(a) + \text{type } 0.$$

For instance, with $\mathrm{N} = 1054$, $a = 484$, $q_1 = 24$, we obtain:

$$\mathrm{Ceil}(25781) = 24 \cdot \mathrm{Ceil}(1054) + \mathrm{Ceil}(484) + \text{type } 0.$$

This concludes the first step of the block decomposition.

Case $n = 2$: After the first mutation at position $q_1\mathrm{N} + a + 1$, the factor becomes:

$$F = \frac{F_a}{2} \cdot (F_\mathrm{N})^{q_1}.$$

We want the next mutation to satisfy:

$$(F_\mathrm{N})^{q_2} \cdot F \geq 2 \quad \text{so} \quad (F_\mathrm{N})^{q_1+q_2} \cdot \frac{F_a^2}{2} \geq 2.$$

Taking logarithms:

$$q_2 = \left\lceil \frac{2(\ln 2 - \ln F_a) - q_1 \ln F_\mathrm{N}}{\ln F_\mathrm{N}} \right\rceil = 23.$$

Hence, the second mutation occurs at:

$$(q_1 + q_2) \cdot \mathrm{N} + 2(a + 1),$$

and the corresponding block decomposition is:

$$\mathrm{Ceil}\left((q_1 + q_2) \cdot \mathrm{N} + 2(a + 1)\right) = q_1 \cdot \mathcal{L}_1 + \mathcal{L}_2 + q_2 \cdot \mathcal{L}_1 + \mathcal{L}_2.$$

In our example: $\mathrm{N} = 1054$, $a = 484$, $q_1 = 24$, $q_2 = 23$, so:

$$\mathrm{Ceil}(50508) = 24 \cdot \mathrm{Ceil}(1054) + \mathrm{Ceil}(484) + 0 + 23 \cdot \mathrm{Ceil}(1054) + \mathrm{Ceil}(484) + 0.$$

Inductive step: Suppose the decomposition holds up to index n:

$$\mathrm{Ceil}\left(\sum_{k=1}^{n} q_k \mathrm{N} + n(a + 1)\right) = \sum_{k=1}^{n} \left(q_k \mathcal{L}_1 + \mathcal{L}_2\right).$$

The residue factor after n steps is:

$$F = \left(\prod_{k=1}^{n} (F_\mathrm{N})^{q_k}\right) \cdot \frac{F_a^n}{2^n}.$$

We want the next mutation to satisfy:

$$F \cdot (F_\mathrm{N})^{q_{n+1}} \cdot \frac{F_a}{2} \geq 2,$$

so:

$$\sum_{k=1}^{n+1} q_k \ln F_{\mathrm{N}} + (n+1)\ln F_a \geq (n+1)\ln 2.$$

Solving gives:

$$q_{n+1} = \left\lceil \frac{(n+1)(\ln 2 - \ln F_a) - \sum_{k=1}^{n} q_k \ln F_{\mathrm{N}}}{\ln F_{\mathrm{N}}} \right\rceil.$$

This completes the inductive proof. □

Remarque 6.5 (On the choice of N)**.** We choose $\mathrm{N} = m + d = 1054$, corresponding to $(m, d) = (665, 389)$, such that:

$$\frac{m}{d} = \frac{665}{389} \approx 1.70951156 > X := \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.70951129.$$

That is, N provides a rational over-approximation of the constant X.

This choice is not unique; any pair (m, d) yielding $\frac{m}{d} > X$ suffices to construct a comparable decomposition. We selected $\mathrm{N} = 1054$ for its numerical precision and reasonable block size.

6.4 Bounding r_n in the Ceil(N) List

Théorème 6.6. *For the transition list* Ceil(N)*, we have:*

$$\frac{m_n}{5} < r_n < m_n, \quad \textit{for all } 0 < n < N.$$

Proof. The proof relies on two complementary arguments depending on the size of N.

1. Direct numerical verification for $N \leq 10^6$ Explicit computation of r_n for $0 < n < 10^6$ confirms the bound numerically. For instance:

$$r_{1000000} = 198875.6767 \approx 0.315 \cdot m_{1000000},$$

which clearly satisfies the inequality.

The entries in the tables below are sorted to highlight:

- Table 1 ($b_n = r_n - m_n$): most negative additive gaps;
- Table 2 ($a_n = r_n/m_n$): smallest multiplicative coefficients;
- Table 3 ($c_n = r_n/m_n$): largest multiplicative coefficients.

Table 1: b_n		Table 2: a_n		Table 3: c_n	
b_n	n	a_n	n	c_n	n
-0.5	1	0.2404498	780239	0.7213476	301994
-0.75	2	0.2404499	478245	0.7213469	603988
-1.344	5	0.2404504	176251	0.7213466	905982
-1.375	3	0.2404506	956490	0.7213466	125743
-1.562	4	0.2404506	654496	0.7213444	427737
-1.762	8	0.2404508	352502	0.7213442	75235
-2.508	7	0.2404516	830747	0.7213440	729731
-2.672	6	0.2404517	528753	0.7213435	251486
-3.321	10	0.2404522	226759	0.7213428	24727
-3.611	13	0.2404524	705004	0.7213421	553480
-3.688	16	0.2404526	403010	0.7213416	855474

These results confirm that:

$$0.24 \cdot m_n < r_n < 0.72 \cdot m_n \quad \text{so} \quad \frac{m_n}{5} < r_n < m_n.$$

2. General asymptotic case for $N > 10^6$ We rely on Lemma 6.4 (block decomposition of Ceil(n)) and on residue accumulation formulas from Section 5.5.

Let:

$$N_0 = \sum_{k=1}^{n} q_k \cdot N + n(a+1) + p,$$

with its decomposition:

$$\mathrm{Ceil}(N_0) = \sum_{k=1}^{n} (q_k \cdot \mathcal{L}_1 + \mathcal{L}_2) + \mathcal{L}_3,$$

where:

- $\mathcal{L}_1 = \text{Ceil}(N)$ is the repeated block,
- $\mathcal{L}_2 = \text{Ceil}(a)$ followed by a type 0 transition,
- $\mathcal{L}_3 = \text{Ceil}(p)$ with $p < N$.

From the additive residue formula, we approximate:

$$r_{N_0} \approx F_0 \cdot \left(\sum_{i=1}^{2n} p_i R_i \right) + R_{2n+1},$$

with typical numerical values:

- $R_1 = r_N \approx 159.99$,
- $R_2 = r_a/2 \approx 73.62$,
- $R_3 = r_p$ bounded independently of n,
- $F_1, F_2 \approx 1$, so $F_0 \approx 1$.

Noting that $R_2 \approx \frac{a+1}{N} \cdot R_1$, we derive:

$$r_{N_0} \approx F_0 \cdot \left(\left(\sum_{k=1}^{n} q_k \right) R_1 + nR_2 \right) + R_3.$$

Using the approximation $R_2 \approx \frac{a+1}{N} \cdot R_1$, we get:

$$r_{N_0} \approx F_0 \cdot \left(\left(\sum_{k=1}^{n} q_k + n \cdot \frac{a+1}{N} \right) R_1 \right) + R_3 = F_0 \cdot \left(\frac{N_0 - p}{N} \cdot R_1 \right) + R_3.$$

The error term $R_3 - F_0 \cdot \frac{p}{N} R_1$ is uniformly bounded (less than 309), and can be neglected asymptotically. We therefore approximate:

$$r_{N_0} \approx F_0 \cdot \frac{N_0}{N} \cdot R_1.$$

Since $m = m_n$ is the total number of type 1 transitions in $\text{Ceil}(N_0)$ and $m_1 = 665$ is the number of such transitions in $\text{Ceil}(N)$, we also have:

$$\frac{N_0}{N} = \frac{m}{m_1}, \quad \text{so} \quad r_{N_0} \approx F_0 \cdot \frac{m}{m_1} \cdot R_1.$$

Bounding F_0: Recall that in the list $\text{Ceil}(N)$, we have

$$m = \left\lceil \frac{\ln 2}{\ln 3} \cdot N \right\rceil, \quad \text{so that} \quad F_0 = \frac{3^m}{2^N}.$$

This implies:

$$\frac{\ln 2}{\ln 3} \cdot N < m \leq \frac{\ln 2}{\ln 3} \cdot N + 1.$$

Hence, there exists a real number $\varepsilon \in (0, 1]$ such that:

$$m = \frac{\ln 2}{\ln 3} \cdot N + \varepsilon.$$

We then express F_0 as:

$$F_0 = \frac{3^m}{2^N} = 3^\varepsilon \cdot \left(\frac{3^{\frac{\ln 2}{\ln 3}}}{2} \right)^N.$$

But $\frac{3^{\frac{\ln 2}{\ln 3}}}{2} = \frac{e^{\ln 2}}{2} = 1$, so this simplifies to:

$$F_0 = 3^\varepsilon.$$

Since $\varepsilon \in (0, 1]$, we conclude:

$$1 < F_0 < 3.$$

Final bounding of r_n: Since $F_0 \in (1,3)$ and $R_1 \approx 159.99$, we write $m = m_n$ for the number of type 1 transitions up to index n, and $m_1 = 665$ for the number of type 1 transitions in $\mathcal{L}_1 = \text{Ceil}(N)$. By the block structure and the approximation established above:

$$r_n \approx F_0 \cdot \frac{m}{m_1} \cdot R_1.$$

Using $F_0 < 3$, we obtain the inequality:

$$\frac{R_1}{m_1} \cdot m < r_n < 3 \cdot \frac{R_1}{m_1} \cdot m.$$

Since

$$\frac{R_1}{m_1} \approx \frac{159.99}{665} \approx 0.2406,$$

we deduce:

$$0.2406 \cdot m < r_n < 0.7218 \cdot m.$$

In particular, for all sufficiently large n, this yields the simpler and uniform bound:

$$\frac{m_n}{5} < r_n < m_n,$$

which confirms the claim. □

7 Study of the Transition List $\text{JGL}(N, v_0)$ (Just Greater List)

Objective. Let $N \in \mathbb{N}^*$ and $v_0 \in \mathbb{N}^*$. We are interested in the set $\text{Up}(N, v_0)$ of transition lists $\mathcal{L}(N, m, d)$ such that v_0 is the minimum of the associated orbit:

$$\text{for all } n \in [1, N], \quad v_n \geq v_0.$$

We define $\text{JGL}(N, v_0)$ as the minimal element of $\text{Up}(N, v_0)$ with respect to the partial order introduced in Section 2.4, namely:

$$\mathcal{L}_1(N, m_1, d_1) \leq \mathcal{L}_2(N, m_2, d_2) \quad \text{if and only if}$$
$$\text{for all } n \in [0, N], \quad m_{n,\mathcal{L}_1} \leq m_{n,\mathcal{L}_2}.$$

This minimum exists because $\text{Up}(N, v_0)$ is finite. It has two principal properties:

- it maximizes r_N among all admissible trajectories with v_0 as minimum;
- it provides a sharp lower bound on the number of type 1 transitions required.

We show that, for a suitable N depending on v_0, one has:

$$\text{JGL}(N, v_0) = \text{Ceil}(N).$$

This identity allows us to:

- determine the minimal cycle length in which v_0 is a strict minimum;
- bound from below the number of type 1 transitions for any list in $\text{Up}(N, v_0)$;
- demonstrate the practical relevance of the approximate sequence v'_n as a classifier for minimal trajectories.

Definition. The list $\text{JGL}(N, v_0)$ is a sequence of N transitions $t_0, t_1, \ldots, t_{N-1}$ satisfying:

- for all $n \in [1, N]$, we have $v_n \geq v_0$, so v_0 is the minimum of the trajectory;
- among all such sequences, transitions of type 0 are prioritized so that, at each step n, the ratio v_n/v_0 is minimized.

We encode $\text{JGL}(N, v_0)$ as a binary word of length N over $\{0, 1\}$, for instance: `1101`.

Remarks.

- Knowing $\text{JGL}(N, v_0)$ determines r_N, the associated residue.
- It maximizes r_N among all lists such that v_0 is the minimal value, since type 1 transitions are deferred as late as possible (see Section 5.4).
- By construction, it is the minimal element of $\text{Up}(N, v_0)$ under the transition-order partial order.

7.1 Step-by-Step Construction of the List $\mathrm{JGL}(N, v_0)$

Assume we know the Syracuse dynamics up to $v_0 \leq \text{maxSyr}$. For $v_0 > \text{maxSyr}$, we construct $\mathrm{JGL}(N, v_0)$ as follows:

Case $N = 1$. We require $v_1 \geq v_0$, which is only possible if $t_0 = 1$ (type 1 transition). Then:

$$v_1 = \frac{3v_0 + 1}{2} = \frac{3^1}{2^1} v_0 + r_1, \quad \text{with } r_1 = \frac{1}{2}.$$

Thus, $\mathrm{JGL}(1, v_0) = \texttt{1}$.

Case $N = 2$. Try $t_1 = 0$, yielding:

$$v_2 = \frac{v_1}{2} = \frac{3}{4} v_0 + \frac{1}{4}.$$

We require $v_2 \geq v_0$ if and only if $v_0 \leq 1$. Therefore:

- if $v_0 = 1$, then $\mathrm{JGL}(2, 1) = \texttt{10}$ (trivial cycle);
- otherwise $t_1 = 1$, and we obtain $\mathrm{JGL}(2, v_0) = \texttt{11}$.

Case $N = 3$. The transition $t_2 = 0$ is admissible since $v_2 > v_0$, hence:

$$v_3 = \frac{v_2}{2} = \frac{3^2}{2^3} v_0 + \frac{5}{8}, \quad \text{so} \quad \mathrm{JGL}(3, v_0) = \texttt{110}.$$

Case $N = 4$. Try $t_3 = 0$:

$$v_4 = \frac{v_3}{2} = \frac{9}{16} v_0 + \frac{5}{16}.$$

To satisfy $v_4 \geq v_0$, we must have $v_0 \leq \frac{5}{7} < \text{maxSyr}$, hence $t_3 = 1$, and:

$$\mathrm{JGL}(4, v_0) = \texttt{1101}.$$

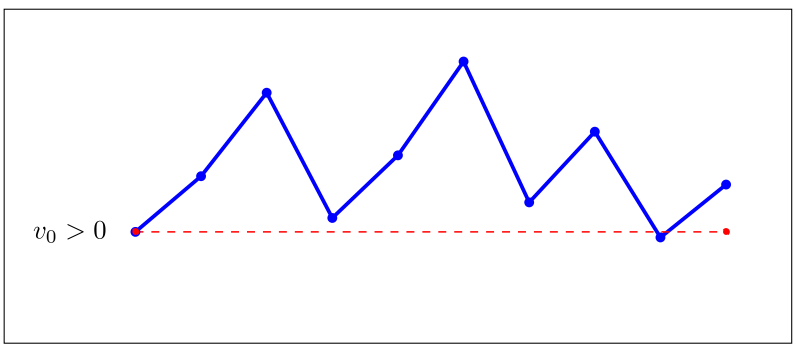

Figure 3: Illustration of the compensation by r_n of dips of v'_n below v_0.

General case. Figure 3 shows how even if $v'_n < v_0$ temporarily, the additive term r_n can compensate for this dip, allowing $v_n \geq v_0$.

Assume $\mathrm{JGL}(N, v_0) = \mathcal{L}(N, m, d)$ has been constructed. Then:

$$v_N = \frac{3^m}{2^N} v_0 + r_N.$$

We test whether $t_N = 0$ is admissible, i.e., whether

$$v_{N+1} = \frac{v_N}{2} \geq v_0 \quad \text{if and only if} \quad v_0 \leq \frac{r_N}{2 - \frac{3^m}{2^N}} =: \mathrm{VMax}(N).$$

- If $v_0 < \mathrm{VMax}(N)$, then $t_N = 0$ is admissible. In that case, $\mathrm{JGL}(N+1, v_0) \neq \mathrm{Ceil}(N+1)$, as the prioritization of type 0 transitions defers a type 1 that Ceil would apply sooner.
- If $v_0 > \mathrm{VMax}(N)$, then $t_N = 1$, and $\mathrm{JGL}(N+1, v_0) = \mathrm{Ceil}(N+1)$.
- If $v_0 = \mathrm{VMax}(N)$:
  - if $N = 1$, we recover the trivial cycle;
  - otherwise, this defines a non-trivial cycle beginning at v_0.

7.2 Construction Tests for the JGL List

We only report the indices n for which $\mathrm{VMax}(n)$ reaches a new maximum (record). These points are critical, as they indicate where $\mathrm{JGL}(n, v_0)$ differs from $\mathrm{Ceil}(n)$ whenever $\mathrm{VMax}(n) > v_0$.

Example test for $N = 20{,}000{,}000$

The maximal difference between the true value r_n and the approximated value $F_n \cdot \dfrac{R_1}{m_1} \cdot m$ is approximately 4942.93, obtained for $n = 19999460$, but the corresponding value is still relatively small in context:

$$r_n = 9{,}099{,}227.8454.$$

k	m	d	n	$\mathrm{VMax}(n)$	$\dfrac{n \ln 2}{\ln\lfloor \mathrm{VMax}(n) \rfloor}$
1	1	0	1	$1 = 2^0$	∞
2	3	1	4	$\approx 2^{2.202}$	2
3	5	2	7	$\approx 2^{4.617}$	1.527
4	17	9	26	$\approx 2^{6.755}$	3.849
5	29	16	45	$\approx 2^{8.139}$	5.532
6	41	23	64	$\approx 2^{9.760}$	6.557
7	94	54	148	$\approx 2^{11.241}$	13.167
8	147	85	232	$\approx 2^{12.247}$	18.943
9	200	116	316	$\approx 2^{13.178}$	23.980
10	253	147	400	$\approx 2^{14.257}$	28.056
11	306	178	484	$\approx 2^{16.137}$	29.993
12	971	567	1538	$\approx 2^{17.865}$	86.092
13	1636	956	2592	$\approx 2^{18.683}$	138.736
14	2301	1345	3646	$\approx 2^{19.244}$	189.463
15	2966	1734	4700	$\approx 2^{19.683}$	238.790
16	3631	2123	5754	$\approx 2^{20.051}$	286.973
17	4296	2512	6808	$\approx 2^{20.374}$	334.154
18	4961	2901	7862	$\approx 2^{20.667}$	380.418
19	5626	3290	8916	$\approx 2^{20.939}$	425.811
20	6291	3679	9970	$\approx 2^{21.197}$	470.354
21	6956	4068	11024	$\approx 2^{21.445}$	514.048
22	7621	4457	12078	$\approx 2^{21.689}$	556.874
23	8286	4846	13132	$\approx 2^{21.931}$	598.794
24	8951	5235	14186	$\approx 2^{22.174}$	639.747
25	9616	5624	15240	$\approx 2^{22.423}$	679.648
26	10281	6013	16294	$\approx 2^{22.682}$	718.373
27	10946	6402	17348	$\approx 2^{22.955}$	755.750
28	11611	6791	18402	$\approx 2^{23.249}$	791.527
29	12276	7180	19456	$\approx 2^{23.573}$	825.334
30	12941	7569	20510	$\approx 2^{23.944}$	856.581
31	13606	7958	21564	$\approx 2^{24.387}$	884.253
32	14271	8347	22618	$\approx 2^{24.955}$	906.349
33	14936	8736	23672	$\approx 2^{25.791}$	917.831
34	15601	9125	24726	$\approx 2^{27.619}$	895.243
35	47468	27766	75234	$\approx 2^{29.960}$	2511.159
36	79335	46407	125742	$\approx 2^{32.277}$	3895.674
37	190537	111456	301993	$\approx 2^{39.369}$	7670.740
38	10781274	6306640	17087914	$\approx 2^{47.594}$	359038.469
39	64497107	37728388	102225495	$\approx 2^{50.657}$	2017986.285
43	6586818670	3853041920	10439860590	$\approx 2^{67.085}$	155621208.2
44	72057431991	42150895612	114208327603	$\approx 2^{71.414}$	1599245808.3

Remarque 7.1 (Growth of the logarithmic coefficient in the record table)**.** The final column in the table, which records the coefficient

$$\frac{n \ln 2}{\ln\lfloor \mathrm{VMax}(n) \rfloor},$$

shows an overall increasing trend as k increases. This coefficient reflects the logarithmic scaling between the index n and the associated maximum value $\mathrm{VMax}(n)$.

The only irregularities in this growth occur at small values of k, specifically $k = 1, 2$, and more significantly at $k = 34$. These exceptions can be attributed to the relatively small size of n, where rounding effects and the limited precision of the approximation

$$X \approx \frac{m}{d+1}$$

can still noticeably affect the outcome. In the case $k = 34$, the value $n = 24{,}726$ is still relatively small in the asymptotic regime, and local fluctuations may impact the logarithmic ratio more strongly.

From that point onward, the values of n increase rapidly. For instance, at $k = 39$, we already have $n = 102{,}225{,}495$, and the size of the underlying records becomes so large that local fluctuations in the approximation are negligible. Consequently, the logarithmic coefficient continues to grow.

Rather than computing further entries in this table—which would require increasingly prohibitive computational effort—we may instead estimate the growth of r_n directly (see Section 6.4) using the approximation $X \approx m/(d+1)$, where X is the limiting constant and the pair (m, d) is derived from the Stern–Brocot method.

As we shall show in Theorem 7.3, the corresponding values of $n = m + d$ are exactly those appearing in the record table. Based on this approximation, one can observe that the coefficient in the last column increases without exception up to $k = 200$, where

$$n = 355{,}531{,}412{,}311{,}100{,}514{,}263{,}425{,}314{,}010{,}019{,}812{,}97,$$

with a corresponding coefficient of approximately

$$1.4394 \times 10^{35}.$$

This illustrates the long-term growth behavior and confirms that each new record in the approximation sequence corresponds to increasingly large values of n, reinforcing the exponential separation in the logarithmic ratio.

7.3 Characterization of the Records of $\mathrm{VMax}(N)$

The values of N for which $\mathrm{VMax}(N)$ reaches a record are precisely those for which $N + 1 = m + d$, where the fraction $\frac{m}{d}$ is a lower rational approximation of the real number

$$X = \frac{\ln 2}{\ln 3 - \ln 2} = \frac{\ln 2}{\ln(3/2)}.$$

These approximations are obtained via the Stern–Brocot tree, which recursively generates all irreducible fractions starting from $\frac{0}{1}$ and $\frac{1}{0}$.

Notation. We denote:

- $\mathrm{app}_k = \frac{m^{(k)}}{d^{(k)}}$ the k^{th} lower approximation of X;
- $n^{(k)} = m^{(k)} + d^{(k)}$;
- A record of $\mathrm{VMax}(N)$ is reached for $N = n^{(k)} - 1$.

Generation method. Initially, set $a = d = 0$ and $b = c = 1$, with $\frac{1}{0}$ representing infinity. At each step, we compare the mediant $\frac{a+c}{b+d}$ of the two bounds $\frac{a}{b}$ and $\frac{c}{d}$ with X, and update one of the bounds depending on whether X is greater or smaller. Only the lower approximations are retained: at each step, a new fraction is recorded only if it is strictly less than X.

Table of the first lower approximations of X Below we give the first values of k for which $\mathrm{app}_k = \frac{m}{d}$, along with the corresponding values of $N = m + d$, and the approximation error $|\frac{m}{d} - X|$ weighted by powers of d.

k	m	d	$N = m + d$	$\frac{m}{d} - X$	$d^2 \cdot$ |**diff**|	$d^3 \cdot$ |**diff**|
1	1	1	2	-7.0951×10^{-1}	7.0951×10^{-1}	7.0951×10^{-1}
2	3	2	5	-2.0951×10^{-1}	8.3805×10^{-1}	1.6761
3	5	3	8	-4.2845×10^{-2}	3.8560×10^{-1}	1.1568
4	17	10	27	-9.5113×10^{-3}	9.5113×10^{-1}	9.5113
5	29	17	46	-3.6289×10^{-3}	1.0488	17.829

k	m	d	$N = m + d$	$\frac{m}{d} - X$	$d^2 \cdot \|\mathbf{diff}\|$	$d^3 \cdot \|\mathbf{diff}\|$
6	41	24	65	-1.1780×10^{-3}	6.7850×10^{-1}	16.2841
7	94	55	149	-4.2038×10^{-4}	1.2717	69.9411
8	147	86	233	-2.0897×10^{-4}	1.5455	132.9139
9	200	117	317	-1.0958×10^{-4}	1.5001	175.5079
10	253	148	401	-5.1832×10^{-5}	1.1353	168.0282
11	306	179	485	-1.4085×10^{-5}	4.5129×10^{-1}	80.7802
12	971	568	1539	-4.2491×10^{-6}	1.3709	778.649
13	1636	957	2593	-2.4094×10^{-6}	2.2067	2111.786
14	2301	1346	3647	-1.6331×10^{-6}	2.9587	3982.4399
15	2966	1735	4701	-1.2049×10^{-6}	3.627	6292.8596
16	3631	2124	5755	-9.3354×10^{-7}	4.2115	8945.2939
17	4296	2513	6809	-7.4619×10^{-7}	4.7123	11841.9915
18	4961	2902	7863	-6.0906×10^{-7}	5.1293	14885.2013
19	5626	3291	8917	-5.0436×10^{-7}	5.4625	17977.172
20	6291	3680	9971	-4.2179×10^{-7}	5.712	21020.1524
21	6956	4069	11025	-3.5500×10^{-7}	5.8777	23916.3914
22	7621	4458	12079	-2.9988×10^{-7}	5.9597	26568.1378
23	8286	4847	13133	-2.5360×10^{-7}	5.9578	28877.6403
24	8951	5236	14187	-2.1419×10^{-7}	5.8723	30747.1477
25	9616	5625	15241	-1.8024×10^{-7}	5.7029	32078.9088
26	10281	6014	16295	-1.5068×10^{-7}	5.4498	32775.1725
27	10946	6403	17349	-1.2471×10^{-7}	5.1129	32738.1875
28	11611	6792	18403	-1.0172×10^{-7}	4.6923	31870.2026
29	12276	7181	19457	-8.1214×10^{-8}	4.1879	30073.4667
30	12941	7570	20511	-6.2818×10^{-8}	3.5998	27250.2285
31	13606	7959	21565	-4.6220×10^{-8}	2.9278	23302.7368
32	14271	8348	22619	-3.1169×10^{-8}	2.1722	18133.2404
33	14936	8737	23673	-1.7459×10^{-8}	1.3327	11643.9881
34	15601	9126	24727	-4.9171×10^{-9}	4.0951×10^{-1}	3737.2287
35	47468	27767	75235	-9.7079×10^{-10}	7.4848×10^{-1}	20783.1259
36	79335	46408	125743	-1.9476×10^{-10}	4.1945×10^{-1}	19465.8376
37	190537	111457	301994	-1.4274×10^{-12}	1.7732×10^{-2}	1976.3845
38	10781274	6306641	17087915	-4.7737×10^{-15}	1.8987×10^{-1}	1197432.5633

Observation. These values of $N-1$ coincide exactly with the indices of the records of VMax($N-1$) observed in Section 7.2 up to $N = 17{,}087{,}915$, which confirms the correspondence with the Stern–Brocot approximations.

Nature of the real number X and behavior of its approximations. For $N = 301994$, the approximation obtained for X corresponds to a particularly accurate fraction, satisfying:

$$\left|\frac{m}{d} - X\right| < \frac{1}{2d^2} = \frac{1.77 \times 10^{-2}}{d^2},$$

which makes it one of the best observed approximations. It coincides with a convergent in the continued fraction expansion of X.

The continued fraction expansion of X begins as follows:

$$[1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, 1, 4, 3, 1, 1, 15, 1, 9, 2, 5, 7, 1,$$
$$1, 4, 8, 1, 11, 1, 20, 2, 1, 10, 1, 4, 1, 1, 1, 1, 1, 37, 4, 55, 1, 1, 49, 1]$$

Although reading these coefficients does not fully characterize the nature of X, one observes that the growth is moderate, with occasional spikes — a typical behavior of "usual" transcendental numbers (in the sense of Baker and Mahler). This contrasts with Liouville numbers, whose coefficients grow rapidly and which admit extraordinarily sharp Diophantine approximations.

Thus, the real number $X = \frac{\ln 2}{\ln(3/2)}$ is assumed to satisfy Roth's lower bound:

$$\left|X - \frac{m}{d}\right| > \frac{C}{d^{2+\varepsilon}}, \quad \text{for all } \varepsilon > 0,$$

which excludes any sequence of too fast approximations. In our case, all observed lower approximations m/d empirically satisfy the bound:

$$X - \frac{m}{d} > \frac{1}{d^\delta}, \quad \text{with } \delta = 3, \text{ except for } d = 1.$$

This observation justifies the use of $\delta = 3$ in later estimates.

However, this is not a formal proof. It cannot be ruled out — depending on the behavior of the convergents at very large scale — that δ might be greater than 3, or even grow asymptotically. Such behavior may be considered in the intended applications of this property.

Property. For any fraction $\frac{m}{d}$ with $N = m + d$, we have the equivalence:

$$1 - \frac{3^m}{2^N} < 0 \quad \text{if and only if} \quad \frac{m}{d} < X,$$

where $X = \dfrac{\ln 2}{\ln 3 - \ln 2}$.

Proof. Let $N = m + d$. We compute:

$$\frac{m}{d} - X = \frac{m}{d} - \frac{\ln 2}{\ln 3 - \ln 2} = \frac{m \ln 3 - N \ln 2}{d(\ln 3 - \ln 2)}.$$

Rewriting the numerator yields:

$$\frac{m \ln 3 - N \ln 2}{d(\ln 3 - \ln 2)} = \frac{\ln(3^m/2^N)}{d(\ln 3 - \ln 2)}.$$

Hence:

$$\frac{3^m}{2^N} < 1 \quad \text{if and only if} \quad \frac{m}{d} < X.$$

□

This equivalence explains why the records of $\mathrm{VMax}(N)$ occur precisely at positions associated with lower rational approximations of X.

Théorème 7.2. *The only values of N for which $\mathrm{VMax}(N)$ reaches a record are those of the form $N = n^{(k)} - 1$, where $\frac{m^{(k)}}{d^{(k)}}$ is a lower rational approximation of $X = \frac{\ln 2}{\ln 3 - \ln 2}$ obtained via the Stern–Brocot tree.*

Proof. **Step 1 – Exhaustive traversal via the JGL algorithm.**

In the construction of $\mathrm{JGL}(N, v_0)$, all pairs (m, d) are explored in increasing order of d, for each m, until the condition

$$\frac{3^m}{2^{N+1}} < 1 \quad \text{that is,} \quad \frac{m}{d+1} < X$$

is met.

This process ensures that all reduced lower approximations of X—i.e., those generated by the Stern–Brocot tree—are eventually encountered, since the algorithm scans all minimal pairs (m, d) with increasing d.

Step 2 – Empirical verification.

Numerical computation for $N \leq 301{,}994$ confirms that each record of $\mathrm{VMax}(N)$ indeed occurs at a value $N = n^{(k)} - 1$, where $n^{(k)} = m^{(k)} + d^{(k)}$ and $\frac{m^{(k)}}{d^{(k)}}$ is a lower approximation of X.

Step 3 – Asymptotic analysis for $N > 301{,}994$.

From Section 6.4, we use the asymptotic expression:

$$r_N \approx \frac{3^m}{2^N} \cdot \frac{m}{m_1} R_1, \quad \text{where } R_1 = 159.98555, \quad m_1 = 665.$$

Hence, the maximal admissible value v_0 (i.e., $\mathrm{VMax}(N)$) satisfies:

$$\mathrm{VMax}(N) \approx \frac{r_N}{2(1 - \frac{3^m}{2^{N+1}})} \approx \frac{3^m}{2^N} \cdot \frac{m}{m_1} \cdot \frac{R_1}{2(1 - \frac{3^m}{2^{N+1}})}.$$

We distinguish three cases based on the quality of approximation $\frac{m}{d+1}$ versus the last known record $\frac{m^{(p)}}{d^{(p)}}$:

- **Case 1:** $\frac{m}{d+1}$ is a better approximation than $\frac{m^{(p)}}{d^{(p)}}$.
Then:
$$\mathrm{VMax}(N) \approx \frac{R_1 X}{m_1(\ln 3 - \ln 2)} \cdot \frac{\frac{3^m}{2^{N+1}}}{X - \frac{m}{d+1}}.$$
Since $\frac{3^m}{2^{N+1}} > \frac{3^{m^{(p)}}}{2^{n^{(p)}}}$ and $X - \frac{m}{d+1} < X - \frac{m^{(p)}}{d^{(p)}}$, we get $\mathrm{VMax}(N) > \mathrm{VMax}(n^{(p)} - 1)$. A new record is thus achieved.

- **Case 2:** $\frac{m}{d+1} = \frac{m^{(p)}}{d^{(p)}}$.
Then $m = km^{(p)}$, $d + 1 = kd^{(p)}$, and
$$\frac{3^m}{2^{N+1}} = \left(\frac{3^{m^{(p)}}}{2^{n^{(p)}}}\right)^k < \frac{3^{m^{(p)}}}{2^{n^{(p)}}},$$
so $\mathrm{VMax}(N) < \mathrm{VMax}(n^{(p)} - 1)$. No record is reached.

- **Case 3:** $\dfrac{m}{d+1}$ is a worse approximation than $\dfrac{m^{(p)}}{d^{(p)}}$, that is:
$$X - \frac{m}{d+1} > X - \frac{m^{(p)}}{d^{(p)}}.$$
Using the identity
$$\frac{3^m}{2^{N+1}} = \exp\left((d+1)(\ln 3 - \ln 2)\left(\frac{m}{d+1} - X\right)\right),$$
we deduce that:
$$X - \frac{m}{d+1} > X - \frac{m^{(p)}}{d^{(p)}} \quad \text{if and only if} \quad \frac{3^m}{2^{N+1}} < \frac{3^{m^{(p)}}}{2^{n^{(p)}}}.$$
Then the value of $\mathrm{VMax}(N)$ is:
$$\mathrm{VMax}(N) = \frac{r_N}{2\left(1 - \frac{3^m}{2^{N+1}}\right)}.$$
Using the approximation $r_N \approx \dfrac{3^m}{2^N} R_1 \cdot \dfrac{m}{m_1}$, we obtain:
$$\mathrm{VMax}(N) \approx \frac{\frac{3^m}{2^N} R_1 \cdot \frac{m}{m_1}}{2\left(1 - \frac{3^m}{2^{N+1}}\right)}.$$
Linearizing the denominator via the approximation $1 - e^{-u} \approx u$ for small $u > 0$, where:
$$u := (d+1)(\ln 3 - \ln 2)\left(\frac{m}{d+1} - X\right),$$
we find:
$$\mathrm{VMax}(N) \approx \frac{R_1 X}{m_1(\ln 3 - \ln 2)} \cdot \frac{\frac{3^m}{2^{N+1}}}{X - \frac{m}{d+1}}.$$
Since $\dfrac{3^m}{2^{N+1}} < \dfrac{3^{m^{(p)}}}{2^{n^{(p)}}}$ and $X - \dfrac{m}{d+1} > X - \dfrac{m^{(p)}}{d^{(p)}}$, both the numerator is smaller and the denominator is larger than for the previous record. Hence:
$$\mathrm{VMax}(N) < \mathrm{VMax}(n^{(p)} - 1),$$
and no record is attained in this case.

Conclusion. Only Case 1 corresponds to the appearance of a new record, which occurs precisely when $\frac{m}{d+1}$ is a new lower approximation of X. $\square$

Conclusion. The only indices N for which $\mathrm{VMax}(N)$ reaches a record correspond exactly to the lower approximations of X in the Stern–Brocot tree. Each record is obtained for $N = n^{(k)} - 1$.

7.4 Theorem: Sufficient Condition on N for $\mathrm{JGL}(N, v_0) = \mathrm{Ceil}(N)$

Théorème 7.3. *Let $k \in \mathbb{N}^*$, and let $v_0 = 2^\alpha$ be a fixed threshold. If:*

$$\mathrm{VMax}(n^{(k-1)} - 1) < v_0 \leq \mathrm{VMax}(n^{(k)} - 1),$$

then for all N such that $0 < N < n^{(k)} - 1$, we have:

$$\mathrm{JGL}(N, v_0) = \mathrm{Ceil}(N).$$

Proof. According to the previous theorem, the only indices N for which $\mathrm{JGL}(N, v_0)$ may differ from $\mathrm{Ceil}(N)$ are those where $\mathrm{VMax}(N) > v_0$. These are precisely the record indices corresponding to lower approximations of X.

If $v_0 > \mathrm{VMax}(n^{(k-1)} - 1)$, then for all $N < n^{(k)} - 1$ that are not record indices, we have $\mathrm{VMax}(N) < v_0$. Thus, the construction of $\mathrm{JGL}(N, v_0)$ exactly matches that of $\mathrm{Ceil}(N)$, which implies the equality of the two lists.

Therefore, no $N < n^{(k)} - 1$ triggers a different transition from that of $\mathrm{Ceil}(N)$, and the equality of transition lists is preserved throughout the interval. □

7.5 Corollary: Minimal Length of a Cycle for a Given Minimum v_0

Corollaire 7.4. *Let $k \geq 1$ and $v_0 = 2^\alpha$ such that:*

$$\mathrm{VMax}(n^{(k-1)} - 1) < v_0 \leq \mathrm{VMax}(n^{(k)} - 1).$$

Then any cycle whose minimal value is v_0 has length $\geq n^{(k)}$, and if such a cycle exists, its minimal length is exactly $n^{(k)}$.

Remark. This result is consistent with the conclusion of Shalom Eliahou (see [4]) regarding minimal cycle lengths, though the approach developed here differs significantly in technique and structure.

Proof. We set $n^{(0)} = 1$ and $\mathrm{VMax}(0) = 0$. For $k = 1$, we retrieve the trivial cycle 1 so 2 so 1, of length 2, corresponding to $n^{(1)} = 2$.

Now assume there exists a nontrivial cycle of length $N > 2$ whose smallest element is $v_0 = 2^\alpha$.

Such a cycle corresponds to a finite sequence $(v_0, v_1, \ldots, v_{N-1})$ satisfying:

- a transition list $\mathcal{L}(N, m, d)$ of length $N = m + d$,
- with v_0 as the minimum along the orbit: i.e., $\mathcal{L}(N, m, d) \in \mathrm{Up}(N, v_0)$.

By construction, the following inequality holds with respect to the partial order defined in Section 2.4:

$$\mathcal{L}(N, m, d) \geq \mathrm{JGL}(N, v_0).$$

We now consider three possible cases based on the number of type 1 transitions:

(1) Case $m < m_{\mathrm{JGL}(N,v_0)}$: This is excluded by the minimality of $\mathrm{JGL}(N, v_0)$ in $\mathrm{Up}(N, v_0)$.

(2) Case $m = m_{\mathrm{JGL}(N,v_0)}$: The transition list has the same number of type 1 transitions. According to Theorem 7.4, as long as $N < n^{(k)}$, we have:

$$\mathrm{JGL}(N, v_0) = \mathrm{Ceil}(N),$$

and since $\mathrm{Ceil}(N)$ produces $v_N > v_0$, this contradicts the periodicity assumption. Therefore, no such cycle can occur for $N < n^{(k)}$.

(3) Case $m > m_{\mathrm{JGL}(N,v_0)}$: In this case, the list has more type 1 transitions, leading to an even faster growth of the orbit. Given that $N < n^{(k)}$ implies $\mathrm{JGL}(N, v_0) = \mathrm{Ceil}(N)$, we also have $m \geq m_{\mathrm{Ceil}(n^{(k)})}$, hence again $N \geq n^{(k)}$.

Therefore, in all cases, any cycle having v_0 as its minimal value must satisfy $N \geq n^{(k)}$. If such a cycle exists, then the smallest possible admissible length is $n^{(k)}$. □

7.6 Theorem: Minimal Growth of c_α for $\alpha \geq 20$

Théorème 7.5. *Define $c_\alpha = n^{(k)} - 1$ for the smallest k such that*

$$\mathrm{VMax}(n^{(k-1)} - 1) < v_0 = 2^\alpha \leq \mathrm{VMax}(n^{(k)} - 1).$$

Then for all $\alpha \geq 20$, we have:

$$c_\alpha > \lceil 285\, \alpha \rceil.$$

Proof. We first examine numerical data from Section 7.2. For $k = 16$, we observe:

$$\mathrm{VMax}(4700) < 2^{20} < \mathrm{VMax}(5754), \quad \text{and} \quad \frac{5754}{20} = 287.7 > 285.$$

This confirms the result for $\alpha = 20$.

Moreover, for k from 16 to 38, and for each worst-case value $v_0 = \lfloor \mathrm{VMax}(n^{(k)} - 1) \rfloor$, the ratio $n^{(k)}/\alpha$ always exceeds 285. For instance, for $v_0 = 1{,}086{,}054$, we compute:

$$\alpha = \log_2(1{,}086{,}054) \approx 20.05, \quad \text{and} \quad \frac{5754}{\alpha} \approx 286.97 > 285.$$

For $k \geq 39$, we enter a range where explicit computation is no longer feasible, and we rely on asymptotic estimates. From Section 6.4, we recall the bound:

$$\mathrm{VMax}(n^{(k)} - 1) \lesssim \frac{R_1 X}{2m_1(\ln 3 - \ln 2)} \cdot \frac{3^{m^{(k)}}}{2^{n^{(k)}}} \cdot \frac{1}{X - \frac{m^{(k)}}{d^{(k)}}},$$

where $X = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.7095$, $R_1 = 159.98555$, and $m_1 = 665$.

Since the approximations $\frac{m^{(k)}}{d^{(k)}}$ converge to X, we may assume:

$$X - \frac{m^{(k)}}{d^{(k)}} \gtrsim \frac{1}{(d^{(k)})^\delta}, \quad \text{with } \delta = 3,$$

as justified in property 7.3.

Moreover, from $n^{(k)} = m^{(k)} + d^{(k)}$ and $\frac{m^{(k)}}{d^{(k)}} \approx X$, we deduce:

$$d^{(k)} \approx \frac{n^{(k)}}{1 + X} \approx 0.3694 \cdot n^{(k)}.$$

Substituting into the previous estimate yields:

$$\mathrm{VMax}(n^{(k)} - 1) \lesssim C \cdot (d^{(k)})^3 \lesssim C' \cdot (n^{(k)})^3,$$

for explicit constants C, C'. Thus, the condition $v_0 = 2^\alpha \leq \mathrm{VMax}(n^{(k)} - 1)$ implies:

$$2^\alpha \lesssim C' \cdot (n^{(k)})^3 \quad \text{so} \quad n^{(k)} \gtrsim 2^{(\alpha + c)/3},$$

for some constant $c \approx \log_2(C') \approx 5.2$.

For instance, for $\alpha = 40$, this gives $n^{(k)} \gtrsim 2^{15} = 32{,}768 \gg 285 \cdot 40 = 11{,}400$.

Conclusion. We have shown that:

- $\mathrm{JGL}(N, v_0) = \mathrm{Ceil}(N)$ for all $N < n^{(k)} - 1$ when $2^\alpha \leq \mathrm{VMax}(n^{(k)} - 1)$;
- and that $n^{(k)} - 1 > \lceil 285\, \alpha \rceil$ for all $\alpha \geq 20$,

thus establishing the theorem.

Remarque 7.6. This proof relies on the lower bound $X - \frac{m}{d} > 1/d^3$, but remains valid for any exponent $\delta > 3$, provided that the bound on α is adjusted accordingly.

□

Remarque 7.7 (Extended remark)**.** In the final application of the Random List Theorem, it is essential that the Syracuse conjecture has been verified up to 2^{α}. The choice $\alpha = 20$ ensures this condition while keeping the bound $c_\alpha > 285\,\alpha$ as small as possible.

However, since the Syracuse conjecture has in fact been verified up to 2^{68}, one could safely take $\alpha = 40$ without requiring any additional assumption. In this case, using the data from Table 6.2 (specifically lines 38 and 39), we observe that:

$$c_\alpha > \lceil 359{,}000\,\alpha \rceil \quad \text{for } \alpha \geq 40.$$

This shows that with $\alpha = 40$, the lower bound obtained in the conclusion of the theorem would be substantially larger than the conservative estimate $c_\alpha > 285\,\alpha$, thereby further reinforcing the robustness and flexibility of the method.

Similarly, we can readily derive the following lower bounds:

$$\begin{array}{lcrl} c_\alpha & > & \lceil \;\; 638\,\alpha \rceil & \text{for } \alpha \geq 22, \\ c_\alpha & > & \lceil 2{,}017{,}000\,\alpha \rceil & \text{for } \alpha \geq 48, \\ c_\alpha & > & \lceil 1{,}599{,}245{,}000\,\alpha \rceil & \text{for } \alpha \geq 68. \end{array}$$

7.7 Upper Bound on the Number of Transition Lists in $\mathrm{Up}(N, v_0)$

Théorème 7.8. *Let* $n_0 = 2^{f(N)}$ *denote the number of transition lists of* $\mathrm{Up}(N, v_0)$*. Then, for all* N *satisfying* $5000 < N \leq c_\alpha$*, one has*

$$\lceil f(N) \rceil \leq 0.953\,N.$$

Proof. Each such list can be interpreted as a discrete path from $(0,0)$ to (d,m) consisting of N elementary steps, where each step is either:

- a horizontal move (type 0 transition), increasing d by 1; or
- a vertical move (type 1 transition), increasing m by 1.

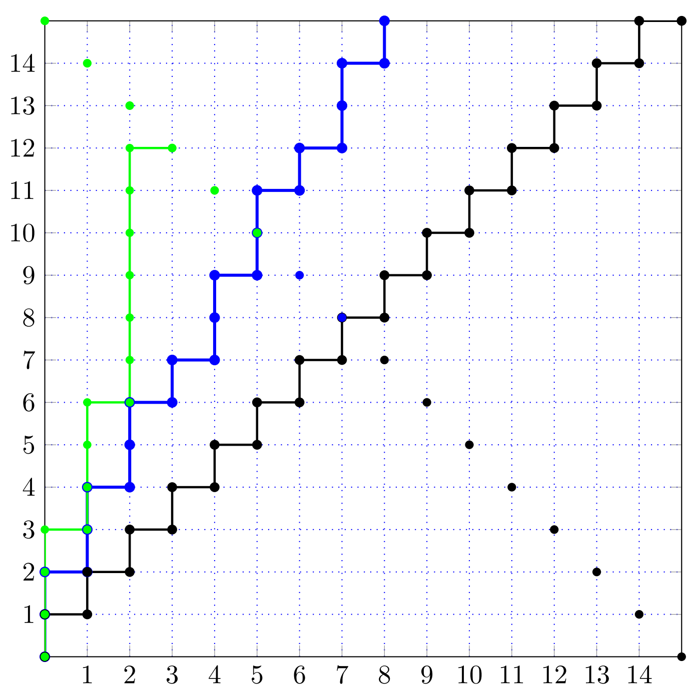

Figure 4: Diagram of transition paths in $\mathrm{Up}(N, v_0)$ relative to the JGL boundary.

In the diagram above:

- The blue line represents the JGL boundary (a constraint to be respected);
- The green line is a valid transition list, always staying above JGL;

- The black line is the classical boundary of the Catalan triangle (without the JGL constraint);
- Green points mark the endpoints of valid transition lists;
- Blue points represent transition lists admissible in the Catalan triangle but invalid under the JGL constraint;
- Black points lie outside both domains.

We aim to upper bound the number $n_0 = 2^{f(N)}$ of transition lists $\mathcal{L}(N, m, d)$ that are greater than or equal to $\mathrm{JGL}(N, v_0)$—that is, to $\mathrm{Ceil}(N)$—for $N \leq c_\alpha$.

To obtain a first upper bound on the number of valid transition lists, we relax the strict constraint imposed by the $\mathrm{JGL}(N, v_0)$ boundary. Instead of requiring each transition list to remain above JGL at every intermediate step (as in the definition of the partial order), we consider the larger set of lists whose endpoint $(m,\, N - m)$ satisfies

$$m \;\geq\; \lceil kN \rceil,$$

where

$$k = \frac{\ln 2}{\ln 3} \approx 0.6309$$

is the lower bound for the number of type-1 transitions obtained earlier.

In other words, we only enforce a condition on the total number of type 1 transitions, without local constraints on the structure of the list.

This is precisely the advantage of having obtained a lower bound for the number of type-1 transitions for $\mathrm{JGL}(N, v_0)$.

We obtain:

$$n_0 < \sum_{m=\lceil kN \rceil}^{N} \binom{N}{m},$$

which is the total number of transition lists with a sufficiently high number of type 1 transitions, regardless of their distribution.

In such a binomial sum centered around its maximum (as here, for m near $kN > N/2$), the dominant term is approximately the largest coefficient. We use the standard approximation (or an elementary upper bound) that the entire sum is at most the number of terms times the maximum coefficient.

Here, the number of terms is at most $N - \lfloor kN \rfloor \leq N$, and the maximum occurs near $m = \lfloor kN \rfloor$. Thus:

$$n_0 < (N - \lfloor kN \rfloor) \cdot \binom{N}{\lfloor kN \rfloor}.$$

For clarity, we also write:

$$n_0 < 0.37\, N \cdot \binom{N}{\lfloor kN \rfloor},$$

using the fact that $1 - k \approx 0.369\ldots$

Using Stirling's approximation (suitably precise for $N > 5000$):

$$\binom{N}{kN} \approx \frac{\sqrt{2\pi N}\,\left(\frac{N}{e}\right)^N}{\sqrt{2\pi kN}\,\left(\frac{kN}{e}\right)^{kN} \cdot \sqrt{2\pi(1-k)N}\,\left(\frac{(1-k)N}{e}\right)^{(1-k)N}}.$$

This simplifies to:

$$\binom{N}{kN} \approx \frac{1}{\sqrt{2\pi k(1-k)N}} \cdot \left(\frac{1}{k^k(1-k)^{1-k}}\right)^N.$$

Taking the logarithm in base 2 gives:

$$f(N) < \log_2(0.37N) - \frac{1}{2}\log_2\left(2\pi k(1-k)N\right) - N\left[k\log_2 k + (1-k)\log_2(1-k)\right].$$

Grouping terms and evaluating numerically with $k = \frac{\ln 2}{\ln 3}$, we obtain:

$$f(N) < f_1(N) = 0.9499556\, N + \tfrac{1}{2}\log_2 N - 1.7089.$$

For $N > 5000$, this yields:

$$f(N) < f_2(N) = 0.9525\, N.$$

since

$$d(N) = f_2(N) - f_1(N) = (0.9525 - 0.9499556)\, N - \tfrac{1}{2}\log_2 N + 1.7089$$

is a strictly increasing function of N for $N > 5000$, and $d(5000) > 2 > 0$.

And finally :

$$\lceil f(N) \rceil \leq 0.953\, N.$$

□

7.8 Conclusion

Summary of results.

- The transition list Ceil(N) is defined by the rule:

$$m_{\mathrm{Ceil}(N),\, n} = \left\lceil \frac{\ln 2}{\ln 3} \cdot n \right\rceil, \quad \text{for all } 0 < n < N.$$

- For any $v_0 = 2^\alpha \in \mathbb{N}$, we denote by $\mathrm{Up}(N, v_0)$ the set of transition lists $\mathcal{L}(N, m, d)$ such that the associated orbit remains greater than or equal to v_0 over N steps.
- The list $\mathrm{JGL}(N, v_0)$ (Just Greater List) is the minimal element of $\mathrm{Up}(N, v_0)$ with respect to the partial order defined in Section 2.4.
- For $N > 5000$, the cardinality of $\mathrm{Up}(N, v_0)$ is less than $2^{0.953\, N}$.
- It follows from Remark 4.3 that the *Random List Theorem* applies to the set of transition lists $\mathrm{Up}(N, v_0)$.

Synthesis of contributions. We have shown that, for each $\alpha \geq 1$, there exists a constant c_α such that for all $N < c_\alpha$, the minimal list $\mathrm{JGL}(N, v_0)$ coincides with Ceil(N). In other words, below this threshold, the optimal transition structure is independent of v_0 and governed solely by the logarithmic proportion $\ln 2/\ln 3$.

This leads to two important consequences:

- an explicit **lower bound** on the number of type 1 transitions in any list $\mathcal{L}(N, m, d) \in \mathrm{Up}(N, v_0)$:

$$m \geq \left\lceil \frac{\ln 2}{\ln 3} \cdot N \right\rceil;$$

- a **constructive criterion** for generating elements of $\mathrm{Up}(N, v_0)$: any list satisfying

$$\mathcal{L}(N, m, d) \geq \mathrm{JGL}(N, v_0)$$

  (in the sense of the cumulative number of type 1 transitions) belongs to $\mathrm{Up}(N, v_0)$.

As a consequence, we have established that the minimal possible length of a cycle whose minimal value is $v_0 = 2^\alpha$ is strictly greater than c_α. This lower bound is obtained without requiring the cyclic condition $v_N = v_0$, but simply from the orbit constraint and the structure of the transition list.

Furthermore, we have proved that :

$$\begin{array}{rcrl} c_\alpha & > & \lceil\ 638\,\alpha\rceil & \text{for } \alpha \geq 22, \\ c_\alpha & > & \lceil 359{,}000\alpha\rceil & \text{for } \alpha \geq 40, \\ c_\alpha & > & \lceil 2{,}017{,}000\,\alpha\rceil & \text{for } \alpha \geq 48, \\ c_\alpha & > & \lceil 1{,}599{,}245{,}000\,\alpha\rceil & \text{for } \alpha \geq 68. \end{array}$$

highlighting the rapid growth of the minimal cycle length as a function of the initial height $v_0 = 2^\alpha$.

Perspective. A central intuition guiding this work is that for large v_0 and moderate n, the exact orbit sequence (v_n) is closely approximated by the idealized sequence (v'_n), due to the boundedness and positivity of the residue r_n. In this regime, the additive corrections induced by the transitions are insufficient to compensate for the addition of an extra type 0 transition. This observation will be key in estimating the asymptotic cardinality of $\mathrm{Up}(N, v_0)$ and justifies the application of the *Random List Theorem*, which underlies the probabilistic part of the argument in the remainder of the proof.

7.9 Empirical Consistency Based on Glide Records

In this section, we test the *Heuristic Approach to Establishing the Existence of Solutions* (see remark 4.4) of the *Random List Theorem* (see 4.2).

We rely here on the data from glide records compiled by Eric Roosendaal (see [8]), which are considered representative of extreme cases. Each value g_k is expressed as 2^n, and we compute the deviation $e = n - N + f(N)$, where $N = N_v$ is the number of steps in the sequence v such that $v_n > v_0$, and $f(N)$ is defined by the relation $\mathrm{Up}(N, g_k) = \mathrm{Ceil}(N) = 2^{f(N)}$.

Notably, all 34 recorded values satisfy the condition $|e| < 6$. Furthermore, the empirical mean of $|e|$ is very close to $\log_2(3)$, indicating that the g_k deviate by only about a factor of 3 on average from the central value predicted by the model (which corresponds to fewer than 2 steps). This stability is remarkable given that these values were not designed to satisfy the theorem's conditions — quite the contrary.

Although these results provide empirical support for a heuristic existence condition (namely when $e \geq -7$) in an extreme regime, they will not be used in the remainder of the main proof of the conjecture. The proof relies solely on formally derived statements derived from the main *Random List Theorem.*

The table below summarizes the data, sorted by increasing values of e.

k	$g_k = 2^n$	n	N_u	N	$\mathrm{Up}(N) = 2^{f(N)}$	$f(N)/N$	e
30	1008932249296231	49.84	1445	886	8.7154561×10^{249}	0.9371	-5.87
3	27	4.76	96	59	1108067472387578	0.8471	-4.26
23	12235060455	33.51	892	547	2.04003975×10^{153}	0.931	-4.20
32	180352746940718527	57.32	1575	966	6.1421750×10^{272}	0.9381	-2.49
18	63728127	25.93	613	376	4.43922580×10^{104}	0.9246	-2.44
19	217740015	27.70	644	395	1.11865878×10^{110}	0.9255	-1.72
26	13179928405231	43.58	1122	688	3.0526516×10^{193}	0.9342	-1.67
27	31835572457967	44.86	1161	712	2.16228481×10^{200}	0.9347	-1.64
4	703	9.46	132	81	1.44591018×10^{21}	0.8678	-1.25
28	70665924117439	46.01	1177	722	1.56857569×10^{203}	0.9349	-0.99
33	1236472189813512351	60.10	1614	990	4.3806510×10^{279}	0.9383	-0.95
22	2788008987	31.38	729	447	6.7791985×10^{124}	0.9277	-0.94
1	3	1.59	6	4	3	0.3962	-0.83
21	1827397567	30.77	706	433	7.2027523×10^{120}	0.9272	-0.75
34	2602714556700227743	61.18	1639	1005	7.880458×10^{283}	0.9384	-0.74
2	7	2.81	11	7	13	0.5286	-0.49
25	2081751768559	40.92	988	606	1.33534773×10^{170}	0.9326	0.06
31	118303688851791519	56.72	1471	902	3.2432244×10^{254}	0.9373	0.18
14	13421671	23.68	468	287	2.2358186×10^{79}	0.9184	0.27
12	1126015	20.10	365	224	3.09780237×10^{61}	0.9119	0.37
20	1200991791	30.16	649	398	7.8702628×10^{110}	0.9256	0.55
16	26716671	24.67	486	298	2.93913359×10^{82}	0.9193	0.62
15	20638335	24.30	476	292	6.0245795×10^{80}	0.919	0.64
6	35655	15.12	220	135	2.12896013×10^{36}	0.8939	0.80
5	10087	13.3	171	105	7.895732×10^{27}	0.8826	0.97
17	56924955	25.76	502	308	2.09716980×10^{85}	0.9202	1.19
24	898696369947	39.71	897	550	1.4304029×10^{154}	0.9311	1.80
9	381727	18.54	282	173	1.13556863×10^{47}	0.9035	1.85
13	8088063	22.95	401	246	5.30115714×10^{67}	0.9145	1.92
7	270271	18.04	267	164	3.36050358×10^{44}	0.9019	1.95
29	739448869367967	49.40	1187	728	7.8482111×10^{204}	0.935	2.03
8	362343	18.47	269	165	6.1418640×10^{44}	0.9017	2.25
10	626331	19.26	287	176	7.8281401×10^{47}	0.904	2.35
11	1027431	19.97	298	183	7.8823885×10^{49}	0.9058	2.72

We obtain:

$$\overline{|e|} = 1.5845\ldots \approx \log_2(3).$$

8 Study of the Lists $\mathrm{JGL}_{2p}(N, v_0)$ for $p > 0$

The list $\mathrm{JGL}(N, v_0)$ captures only the constraint that the initial value v_0 remains the minimum over the first N steps of an orbit.

However, in order to apply the *Random List Theorem* (see Section 4), it is necessary to identify at least 30 solutions of the considered problem.

The purpose of the lists $\mathrm{JGL}_{2p}(N, v_0)$ is to ensure that the first $p+1$ values of the orbit are solutions of the sample.

Throughout this section, we restrict ourselves to values of v_0 that are indeed minimal for some trajectory in $\mathrm{Up}(N, v_0)$, i.e., satisfying $v_0 = \min\{v_n \mid 0 \le n \le N\}$.

8.1 Definition of the List $\mathrm{JGL}_{2p}(N, v_0)$

Let $N \in \mathbb{N}^*$, $p \in \mathbb{N}$ with $2p \le N$, and suppose $v_0 = \min\{v_n \mid 0 \le n \le N\}$.

We define the list $\mathrm{JGL}_{2p}(N, v_0)$ by

$$\mathrm{JGL}_{2p}(N, v_0) := \underbrace{00\cdots0}_{2p \text{ transitions}} \ \mathrm{JGL}(N-2p, v_0).$$

Remarque 8.1.

- The notation represents a binary word: the list begins with $2p$ type 0 transitions, followed by the $N-2p$ transitions from $\mathrm{JGL}(N-2p, v_0)$.
- For $p = 0$, the definition reduces to the classical case:

$$\mathrm{JGL}_0(N, v_0) = \mathrm{JGL}(N, v_0).$$

- By construction, this list satisfies for all $1 \le n \le N$ the inequality:

$$v_n \ge \frac{v_0}{2^{2p}}, \qquad \text{or equivalently,} \qquad \frac{v_n}{v_0} \ge \frac{1}{2^{2p}}.$$

8.2 Definition of the Set $\mathrm{Up}_{2p}(N, v_0)$

We denote by $\mathrm{Up}_{2p}(N, v_0)$ the set of transition lists $\mathcal{L}(N, m, d)$ satisfying

$$\mathcal{L}(N, m, d) \ge \mathrm{JGL}_{2p}(N, v_0)$$

under the partial order defined in Section 2.4. This means that at each step n, the cumulative number of type 1 transitions does not fall below that of $\mathrm{JGL}_{2p}(N, v_0)$.

8.3 Dynamical Comparison of Shifted Orbit Values

Théorème 8.2. *Let $\{v_n\}$ be an orbit such that $v_0 = \min\{v_k : 0 \le k \le N\}$. Then, for all $0 < n < N$,*

$$\frac{v_{N+n}}{v_n} \ge \frac{1}{2^{2n}} \cdot \frac{v_N}{v_0}, \quad \textit{and thus} \quad v_{N+n} \ge \frac{1}{2^{2n}} \cdot v_n.$$

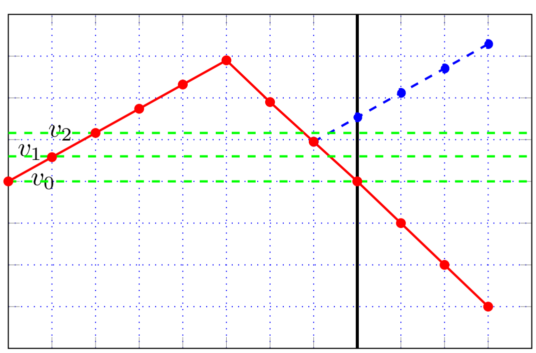

Figure 5: Illustration of the theorem. The black line marks index N; the red curve depicts a worst-case trajectory, while the green dashed lines highlight the comparison between v_n and v_{N+n}.

Diagram Commentary:

- The black vertical line corresponds to the index N;
- The red trajectory illustrates the extreme case where $v_0 < v_n$ for all $0 < n < N$, with initial transitions of type 1, and $v_0 > v_{N+n}$ for small values of n;
- The green dotted lines help to compare v_0, v_1, v_2 with v_N, v_{N+1}, v_{N+2} respectively.

Note: All segments connecting v_n to v_{N+n} are of length N, for every n.

Proof. The result is proven by induction on $n \in \mathbb{N}^*$, with $n < N$.

Base Case ($n = 1$): We have:

$$v_1 \leq \frac{3v_0 + 1}{2} \leq 2v_0 \quad \text{(since } v_0 \geq 1\text{)},$$

thus:

$$\frac{v_0}{v_1} \geq \frac{1}{2}.$$

Also,

$$v_{N+1} \geq \frac{v_N}{2} \qquad \text{so} \qquad \frac{v_{N+1}}{v_N} \geq \frac{1}{2}.$$

Combining:

$$\frac{v_{N+1}}{v_1} = \frac{v_{N+1}}{v_N} \cdot \frac{v_N}{v_0} \cdot \frac{v_0}{v_1} \geq \frac{1}{2} \cdot \frac{v_N}{v_0} \cdot \frac{1}{2} = \frac{1}{4} \cdot \frac{v_N}{v_0}.$$

Thus the case $n = 1$ is verified.

Inductive Step: Suppose the result holds for some $n < N$, that is:

$$\frac{v_{N+n}}{v_n} \geq \frac{1}{2^{2n}} \cdot \frac{v_N}{v_0}.$$

We aim to show it holds for $n + 1$.

We have:

$$v_{n+1} \leq \frac{3v_n + 1}{2} \leq 2v_n \qquad \text{so} \qquad \frac{v_n}{v_{n+1}} \geq \frac{1}{2},$$

and:

$$v_{N+n+1} \geq \frac{v_{N+n}}{2} \qquad \text{so} \qquad \frac{v_{N+n+1}}{v_{N+n}} \geq \frac{1}{2}.$$

Therefore:

$$\frac{v_{N+n+1}}{v_{n+1}} = \frac{v_{N+n+1}}{v_{N+n}} \cdot \frac{v_{N+n}}{v_n} \cdot \frac{v_n}{v_{n+1}} \geq \frac{1}{2} \cdot \left(\frac{1}{2^{2n}} \cdot \frac{v_N}{v_0} \right) \cdot \frac{1}{2} = \frac{1}{2^{2(n+1)}} \cdot \frac{v_N}{v_0}.$$

Thus, the property holds for $n + 1$. By induction, it holds for all $0 < n < N$. □

8.4 Corollary — Uniform Bound on Shifted Values

Corollaire 8.3. *Let $N \in \mathbb{N}^*$, $v_0 = \min\{v_n \mid 0 \leq n \leq N\}$, and $0 < p < N$. Then, for all $0 \leq n \leq N$,*

$$v_{p+n} \geq \frac{1}{2^{2p}} \cdot v_p.$$

Proof. We proceed by induction on p.

Base Case ($p = 1$): For any $0 \leq n \leq N$, we distinguish two cases.

- If $0 < n < N$, then $n + 1 \leq N$ and thus $v_{1+n} \geq v_0$ (since $v_0 = \min\{v_k : 0 \leq k \leq N\}$). As $v_1 \leq 2v_0$ (from the previous proof), we have:

$$v_{1+n} \geq \frac{v_1}{2} \geq \frac{1}{4} \cdot v_1.$$

- If $n = N$, then $v_{1+N} \geq \frac{1}{4} \cdot \frac{v_N}{v_0} \cdot v_1 > \frac{1}{4} \cdot v_1$ by Theorem 7.1 (as $v_N/v_0 \geq 1$).

So the property holds for $p = 1$.

Inductive Step: Assume the property holds for some p with $0 < p < N$. We want to prove it for $p + 1$.

For $0 \le n \le N$, the inductive hypothesis gives:

$$v_{p+n} \ge \frac{1}{2^{2p}} \cdot v_p.$$

Since $v_{p+n+1} \ge v_{p+n}/2$ and $v_{p+1} \le 2v_p$, we have:

$$v_{p+n+1} \ge \frac{1}{2} \cdot \left(\frac{1}{2^{2p}} \cdot v_p \right) = \frac{1}{2^{2p+1}} \cdot v_p \ge \frac{1}{2^{2(p+1)}} \cdot v_{p+1}.$$

Thus, the property is true at step $p + 1$. By induction, it holds for all $0 < p < N$. $\square$

8.5 Sufficient Condition on N to Ensure $\mathrm{JGL}_{2p}(N, v_0) = 0^{2p}\mathrm{Ceil}(N - 2p)$

Théorème 8.4. *Let $k \in \mathbb{N}^*$, and let $v_0 = 2^\alpha$ be a fixed threshold. If*

$$\mathrm{VMax}(n^{(k-1)} - 1) < v_0 \le \mathrm{VMax}(n^{(k)} - 1),$$

then for every N such that $0 < N < n^{(k)} - 1$, we have:

$$\mathrm{JGL}_{2p}(N, v_0) = 0^{2p}\,\mathrm{Ceil}(N - 2p),$$

and the number of type 1 transitions satisfies:

$$m_{\mathrm{JGL}_{2p}(N,v_0)} = \left\lceil \frac{\ln 2}{\ln 3} \cdot (N - 2p) \right\rceil.$$

Proof. By definition of $\mathrm{JGL}_{2p}(N, v_0)$, we have:

$$\mathrm{JGL}_{2p}(N, v_0) := 0^{2p} \cdot \mathrm{JGL}(N - 2p, v_0).$$

Under the given assumption, Theorem 7.4 ensures that for all $0 < N - 2p < n^{(k)} - 1$, we have:

$$\mathrm{JGL}(N - 2p, v_0) = \mathrm{Ceil}(N - 2p).$$

Hence, we conclude:

$$\mathrm{JGL}_{2p}(N, v_0) = 0^{2p} \cdot \mathrm{Ceil}(N - 2p).$$

Finally, since $\mathrm{Ceil}(N - 2p)$ contains exactly

$$\left\lceil \frac{\ln 2}{\ln 3} \cdot (N - 2p) \right\rceil$$

type 1 transitions, the list $\mathrm{JGL}_{2p}(N, v_0)$ contains the same number. $\square$

8.6 Upper Bound on the Number of Transition Lists in $\mathrm{Up}_{2p}(N, v_0)$

Théorème 8.5. *Let $n_0 = 2^{f(N)}$ denote the number of transition lists of $\mathrm{Up}_{2p}(N, v_0)$. Then, for all N satisfying $14000 < N \le c_\alpha$ and $200 < p < 0.05\,N$, one has*

$$\lceil f(N) \rceil \le 1.2\,p + 0.95\,N.$$

Proof. We want to upper bound the cardinality $n_0 = 2^{f(N)}$ of the set $\mathrm{Up}_{2p}(N, v_0)$. To this end, we consider lists $\mathcal{L}(N, m, d)$ satisfying only the final constraint $m \ge \lceil k(N - 2p) \rceil$, where $k = \frac{\ln 2}{\ln 3} \approx 0.6309$. This constraint is weaker than requiring the entire trajectory to remain above $\mathrm{JGL}_{2p}(N, v_0)$ pointwise, but suffices for an effective upper bound.

We observe the inequality:

$$\lceil k(N - 2p) \rceil > kN - 1.262p > \lceil kN \rceil - \tfrac{3}{2}p.$$

Thus, we upper bound the cardinality of $\mathrm{Up}_{2p}(N, v_0)$ as:

$$n_0 < \sum_{m=\lceil kN\rceil - \frac{3}{2}p}^{N} \binom{N}{m}.$$

We bound each term in the sum by the largest among them, attained for $m = \lceil kN\rceil - \frac{3}{2}p$, which is valid as long as $m > \frac{N}{2}$ — a condition satisfied whenever $N \geq 22 \times 638 > 14{,}000$ and $p \leq 0.05\,N$.

The number of terms in the sum is bounded by:

$$N - \lfloor kN\rfloor + \tfrac{3}{2}p < 0.37\,N + \tfrac{3}{2}p < 0.5\,N.$$

To bound $\binom{N}{kN-\frac{3}{2}p}$, we use an estimate based on the ratio between this shifted coefficient and the central binomial coefficient $\binom{N}{kN}$. Using the recurrence formula for binomial coefficients:

$$\binom{N}{kN - \frac{3}{2}p} = \prod_{i=0}^{\frac{3}{2}p-1} \frac{kN - i}{N - kN + i + 1} \cdot \binom{N}{kN}.$$

Each factor in the product is at most $\frac{kN}{N-kN}$, so:

$$\binom{N}{kN - \frac{3}{2}p} < \left(\frac{k}{1-k}\right)^{\frac{3}{2}p} \cdot \binom{N}{kN}.$$

Hence, the total number of admissible lists satisfies:

$$n_0 < 0.5\,N \cdot \left(\frac{k}{1-k}\right)^{\frac{3}{2}p} \cdot \binom{N}{kN}.$$

We apply Stirling's approximation for $\binom{N}{kN}$, as in Method 1:

$$\begin{aligned} f(N) <\; & \log_2(0.5N) + \frac{3}{2}p\log_2\left(\frac{k}{1-k}\right) \\ & - \frac{1}{2}\log_2\left(2\pi k(1-k)N\right) - N\left[k\log_2 k + (1-k)\log_2(1-k)\right]. \end{aligned}$$

By grouping constants and factoring terms:

$$\begin{aligned} f(N) <\; & \log_2(0.5) - \frac{1}{2}\log_2\left(2\pi k(1-k)\right) + \frac{3}{2}p\log_2\left(\frac{k}{1-k}\right) \\ & + \frac{1}{2}\log_2(N) - N\left[k\log_2 k + (1-k)\log_2(1-k)\right]. \end{aligned}$$

After numerical evaluation we obtain

$$f(N) < -1.2745 + 1.1604\,p + \tfrac{1}{2}\log_2 N + 0.9499556\,N.$$

For convenience, let us introduce the functions

$$f_1(N,p) = -1.2745 + 1.1604\,p + \tfrac{1}{2}\log_2 N + 0.9499556\,N, \quad f_2(N,p) = 1.2\,p + 0.95\,N.$$

We then consider their difference

$$d(N,p) = f_2(N,p) - f_1(N,p) = 1.2745 + (1.2 - 1.1604)p - \tfrac{1}{2}\log_2 N + (0.95 - 0.9499556)N$$

This function is increasing in both p and N for $N > 14{,}000$. Moreover,

$$d(14{,}000; 200) = 1.2745 + 0.0396 \times 200 - \tfrac{1}{2}\log_2(14{,}000) + 0.0000444 \times 14{,}000 > 2.9 > 0.$$

Hence, whenever $N \geq 22 \times 638 > 14{,}000$ and $p > 200$, the following upper bound holds:

$$\lceil f(N)\rceil < 1.2\,p + 0.95\,N.$$

□

8.7 Conclusion

According to the corollary established in Section 8.4, if v_0 is the minimal value along an orbit associated with a transition list in the set $\mathrm{Up}(N, v_0)$, then at least the first $p+1$ values of the orbit—namely $v_0, v_1, \ldots, v_p$—satisfy the following properties:

- For every $0 \leq n \leq p$, the value v_n is itself a minimal value for a transition list belonging to the set $\mathrm{Up}_{2p}(N, v_0)$, as defined in Section 8.4;
- In particular, each of these values lies in the interval:

$$v_n \in [v_0,\ 2^p v_0[\quad \text{for all } 0 \leq n \leq p.$$

This ensures that the set $\mathrm{Up}_{2p}(N, v_0)$ effectively captures at least the first $p+1$ distinct solutions of the associated problem.

Moreover, for all thresholds $v_0 = 2^\alpha$ with $\alpha \geq 20$, and for every $N \leq c_\alpha$, the sufficient condition of Theorem 7.4 remains satisfied. As a consequence, the list $\mathrm{JGL}_{2p}(N, v_0)$ coincides with the shifted version of the canonical list:

$$\mathrm{JGL}_{2p}(N, v_0) = 0^{2p}\,\mathrm{Ceil}(N - 2p).$$

This equality confirms that, up to $N \leq c_\alpha$, the structural behavior of the filtered lists JGL_{2p} remains entirely predictable and independent of v_0.

Moreover, we have shown that if $n_0 = 2^{f(N)}$ denotes the number of transition lists of $\mathrm{Up}_{2p}(N, v_0)$, then for all N satisfying $14{,}000 < N \leq c_\alpha$ and $200 < p < 0.05\,N$, one has

$$\lceil f(N) \rceil \leq 1.2\,p + 0.95\,N.$$

Furthermore, by Remark 4.3, the *Random List Theorem* also applies to the set $\mathrm{Up}_{2p}(N, v_0)$.

9 Proof of the Collatz Conjecture

Remarque 9.1 (Equivalent Statements of the Collatz Conjecture)**.** The Collatz conjecture (also known as the $3x + 1$ conjecture or the Syracuse problem) admits several equivalent formulations, each highlighting a different aspect of the conjectured dynamical behavior.

The conjecture is equivalent to the following three statements:

(1) (Convergence to 1)
For every $v_0 > 1$, there exists an integer $n \geq 0$ such that $v_n = 1$.

(2) (Strict Descent)
For every $v_0 > 1$, there exists $n \geq 1$ such that $v_n < v_0$.

(3) (Absence of Non-Trivial Cycles and Divergence)
The sequence (v_n) admits no cycles other than the trivial cycle $\{1, 2\}$, and there exists no v_0 such that (v_n) diverges (i.e., tends to $+\infty$).

Each of these statements captures a key facet of the expected behavior of the sequence (v_n): statement (1) expresses convergence, (2) ensures the absence of strictly stationary trajectories above 1, and (3) excludes both non-trivial periodic behavior and divergence. Their equivalence is well established in the literature (see, for example, Lagarias [7]).

Proof. The Collatz conjecture has been verified numerically for all initial values $v_0 \leq 2^{68}$ as of January 1st, 2025.

We propose an inductive proof, based on the following three statements, assumed true for all values strictly less than $v_0 = 2^\alpha$:

(i) There exists no non-trivial cycle whose minimum value is strictly less than v_0.

(ii) No orbit starting from a value $< v_0$ diverges.

(iii) No orbit starting at $W_0 \leq v_0$ reaches a cycle whose minimum is v_0.

The argument relies on the *Random List Theorem* (see Section 4) and ultimately reduces to a counting problem.

Base case. The conjecture holds for all $v_0 \le 2^{\alpha_0}$ with $\alpha_0 \le 68$.

Several values of $\alpha_0 \in \{20, 22, 48, 68\}$ will be used throughout the proof, depending on the context. Thus, assertions **(i)–(iii)** are satisfied at this level.

Inductive step at $v_0 = 2^\alpha$. Let $\alpha > \alpha_0$. We assume the validity of **(i)–(iii)** for all values $< v_0$, and aim to establish them at v_0.

A possible counterexample must fall into one of the two following categories:

(a) A non-trivial cycle whose minimal value is exactly v_0;

(b) A divergent orbit beginning at v_0.

By the induction hypothesis, no cycle or divergence can occur for a value $< v_0$.

Let $N = c_{\alpha_0}$. We will prove that v_0 cannot belong to a trajectory in $\mathrm{Up}(N, v_0)$, which suffices to exclude both cases (a) and (b), because:

- Any cycle with minimum value v_0 must have length strictly greater than N;
- Any divergent orbit starting from v_0 would need to belong to $\mathrm{Up}(N, v_0)$ for all N.

Assertion (iii) is particularly important: we must ensure that the initial value is indeed the minimum over the first N terms of the orbit. If some $W_0 < v_0$ led to a cyclic orbit with v_0 as its minimum, we would need to exclude the corresponding transition lists, as they would represent neither a length-N cycle nor divergence; yet W_0 would be the minimum of its orbit, invalidating the reasoning.

Such a situation does not occur here: the value 1 is the smallest possible and is the minimum of the trivial cycle $\{1, 2\}$. No transition list needs to be excluded. Thus, assertion (iii) is automatically verified as long as no additional cycle exists.

As an illustration, consider the extended variant $v_{n+1} = \frac{3v_n+5}{2}$ for odd v_n. In this case, the orbit of $v_0 = 3$ reaches the cycle $\{19, 31, 49, 76, 38\}$ even though $3 < 19$. This orbit is called pre-cyclic, and the corresponding transition lists must be excluded. Nothing similar occurs in our setting.

Since $N > \alpha$, we are only concerned with *minimal solutions* of the trajectories. Any non-minimal solution must be built from a minimal one, and its value is necessarily greater than 2^N.

- **Method 1:** Based on the structure of the lists $\mathrm{JGL}(N, v_0)$ (see Section 7.8). The cycle or divergence is identified by the minimal value v_0. We apply the *Heuristic Approach to Establishing the Existence of Solutions* (see 4.4) to show that no solution is possible.

  In this case, we choose $\alpha_0 = 20$, and then $c_\alpha \ge \lceil 285\,\alpha \rceil$ and $\lceil f(N) \rceil < 0.953\,N$.
  We set $n = \alpha$ (since $v_0 = 2^\alpha$), and compute the gap:

  $$e = n - N + \lceil f(N) \rceil < \alpha - 285\,\alpha + 0.953 \cdot 285\,\alpha = -12.395\,\alpha.$$

  As soon as $\alpha > 20$, we get $e < -247.9 \ll -7$, and thus the heuristic approach implies that no solution exists. Although this method is heuristically well motivated, it lacks a full theoretical justification.
  Assuming use of the heuristic approach, no trajectory starting from $v_0 = 2^\alpha$ is compatible with the existence of a non-trivial cycle or a divergent orbit. This suffices to validate the three assertions of the induction hypothesis at rank v_0.
  This method relies solely on a property that has been numerically verified, but not formally proven, and therefore carries no mathematical weight in a strict sense. However, the encouraging results it yields serve as a motivation and foundation for the following method, which is mathematically rigorous.

- **Method 2:** Based on the lists $\mathrm{JGL}_{2p}(N, v_0)$ (see Section 8.7). Here, the cycle or divergence is identified by the p values $v_0, v_1, \ldots, v_p$ of the orbit for which v_0 is the minimum. The set $\mathrm{Up}_{2p}(N, v_0)$ is considered, and we show that the number of possible solutions is strictly less than $p+1$, which leads to a contradiction. This method lies on *the Random List Theorem* and thus constitutes a fully satisfactory proof from a formal standpoint. Several values of p and α_0 will be used throughout the proof, depending on which part (CLT or Berry–Esseen) of the theorem is applied and on the desired probability threshold required to obtain a contradiction.

  We will use values of $\alpha_0 \in \{22, 48, 68\}$ to ensure prior verification of the conjecture up to 2^{α_0}.
  For $v_0 = 2^\alpha$, depending on the chosen value of α_0, we will select values of N equal to the corresponding lower bounds of c_α, namely $\lceil 638\,\alpha \rceil$, $\lceil 2{,}017{,}000\,\alpha \rceil$, and $\lceil 1{,}599{,}245{,}000\,\alpha \rceil$.

We fix $p \leq 0.05N$ and consider the first $p+1$ values of the orbit associated with the first $\text{N} = c_\alpha$ steps. Since each v_i satisfies $v_i \in [v_0, 2^p v_0[$ for $0 \leq i \leq p$, we apply the theorem with $n = \alpha + p$, accounting for the logarithmic scale (base 2). Under these conditions, we expect to find at least $p+1$ minimal solutions belonging to transition lists in the set $\text{Up}_{2p}(N, v_0)$.

Application of the Random List Theorem :

We compute:

$$e = n - N + \lceil f(N) \rceil,$$

and we have the following upper bound:

$$e < n + 1.2\,p - 0.05\,N.$$

- **Using the Central Limit Theorem (CLT):**
  Let $z = 4$, so that $\varepsilon = 1 - \Phi(4) \approx 3.2 \times 10^{-5}$.
  According to the theorem, if $e \leq 6$, then

  $$R_n < 64 + 8 \cdot z = 96,$$

  with probability at least $1 - \varepsilon$.
  To obtain a contradiction, we compute e for a value of $p+1$ significantly larger than 96, aiming to reach a regime where $e \ll 6$.
  * For $\alpha_0 = 22$, we apply the theorem with

    $$\alpha \geq 22, \quad n = \alpha + p, \quad N = \lceil 638\,\alpha \rceil, \quad 200 < p = 300 < 0.05 \times N.$$

    Substituting into the expression for e, we obtain an upper bound that is strictly decreasing in α:

    $$e < 2.2\,p - 30.9\,\alpha < 660 - 679.8 = -19.8 < 6.$$

    According to the *Random List Theorem*, such a value of e implies that the number of solutions is strictly less than 96. However, to generate a cycle or divergence, we would require at least $p + 1 = 301$ minimal solutions below 2^n.
    This leads to a contradiction: under the binomial distribution governing R_n, we cannot have more than 96 values with probability greater than $1 - \varepsilon$, yet we require at least 301.
    Therefore, we conclude that the recurrence hypothesis holds at level $v_0 = 2^\alpha$.
  * For $\alpha_0 = 48$, we apply the theorem with

    $$\alpha \geq 48, \quad n = \alpha + p, \quad N = \lceil 2{,}017{,}000\,\alpha \rceil, \quad p = 2{,}000{,}000 < 0.05\,N.$$

    Substituting into the expression for e, we obtain an upper bound that is strictly decreasing in α:

    $$e < 2.2\,p - 100{,}849\,\alpha < 4{,}400{,}000 - 4{,}840{,}752 = -440{,}752 \ll 6.$$

    This again results in a contradiction, as the number of required solutions greatly exceeds the theoretical upper bound: $p + 1 = 2{,}000{,}001 \gg 96$.
    The discrepancy is sufficiently large to allow for a safe and reliable application of the Berry–Esseen inequality.
- **Using the Berry–Esseen inequality:**
  * For $\alpha_0 = 48$, the previous computation with $p = 2{,}000{,}000$ yielded a value of $e < -440{,}752 \ll 6 < 20$.
    Given that $2{,}000{,}001 \gg 1{,}052{,}383$, the *Random List Theorem*, together with the binomial distribution governing R_n, yields a formal contradiction — rigorously confirmed via the Berry–Esseen inequality with probability at least $1 - \varepsilon$, where $\varepsilon = 10^{-3}$.
  * For $\alpha_0 = 68$, we apply the theorem with

    $$\alpha \geq 68, n = \alpha + p, N = \lceil 1{,}599{,}245{,}000\alpha \rceil, p = 2{,}000{,}000{,}000 < 0.05N.$$

    Substituting into the expression for e, we obtain an upper bound that is strictly decreasing in α:

    $$e < 2.2\,p - 79{,}962{,}248\,\alpha < -1{,}037{,}432{,}864 \ll 6 < 26.$$

Given that 2,000,000,001 $\gg$ 67,144,024, the *Random List Theorem*, together with the binomial distribution governing R_n, yields a formal contradiction — rigorously confirmed via the Berry–Esseen inequality with probability at least $1 - \varepsilon$, where $\varepsilon = 10^{-4}$. Therefore, by a proof by contradiction, we conclude that the recurrence hypothesis must hold at level $v_0 = 2^\alpha$.

In this Method 2, the application of the theorem ensures complete formal justification.

Ultimately, verifying the conjecture up to 2^{48} is sufficient to obtain a formal contradiction, even when using the Berry–Esseen inequality.

□

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