Conjecture of Syracuse

Key steps

of the proof of

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Full proof available on the website

https://www.bajaxe.com

Standard Syracuse sequence : u

$$\begin{cases} u_0 > 0 \\ u_{n+1} = u_n/2 \quad \text{if} \quad u_n \quad \text{is} \quad \text{even (transition of type 0)} \\ u_{n+1} = 3u_n + 1 \quad \text{if} \quad u_n \quad \text{is} \quad \text{odd (transition of type 1)} \end{cases}$$

Reduced Syracuse sequence : v

As if u_n is odd, u_{n+1} is even by construction, it is interesting to make the following transition directly.
The reduced Syracuse sequence brings together this transition.

$$\begin{cases} v_0 > 0 \\ v_{n+1} = v_n/2 \quad \text{if} \quad v_n \quad \text{is} \quad \text{even (transition of type 0)} \\ v_{n+1} = (3v_n + 1)/2 \quad \text{if} \quad v_n \quad \text{is} \quad \text{odd (transition of type 1)} \end{cases}$$

Remark : The type of transition corresponds to the value of the least significant bit (bit 0) of v_n. Bit 0 contains the parity information of v_n.

Remark : In the following, we will focus solely on the sequence v

Conjecture of Syracuse

Statement : For any value of v_0, there exists n such that $v_n = 1$

Remark : The conjecture has been verified for $v_0 \leq 2^{68}$; for the proof, a verification up to 2^{28} is sufficient.

Main idea: Instead of focusing on the orbit generated from v_0 (the set of values v_n), which is chaotic and for which no mathematical modeling is suitable, we will study the statistical distribution of minimal solutions for a set of random transition lists (the set of ”transition type” or ”parity vector”) using standard and robust mathematical tools.

Result: The apparent chaos transforms into a continuum.

Transition list $L(N, m, d)$ for v

Definition: A transition list is a word composed of 0s and 1s corresponding to the type of each transition.

We set:

- d as the number of "type 0" transitions ("divisions")
- m as the number of "type 1" transitions ("multiplications")
- $N = m + d$

We denote by m_n or $m_{n,L}$ the number of "type 1" transitions in the sublist of the first n transitions of a list $L(N\,, m\,, d)$. Similarly, we denote by d_n or $d_{n,L}$ the number of "type 0" transitions in the sublist of the first n transitions of a list $L(N\,, m\,, d)$.

Partial order: We define a partial order relation on the transition lists $L(N\,, m\,, d)$ as follows:

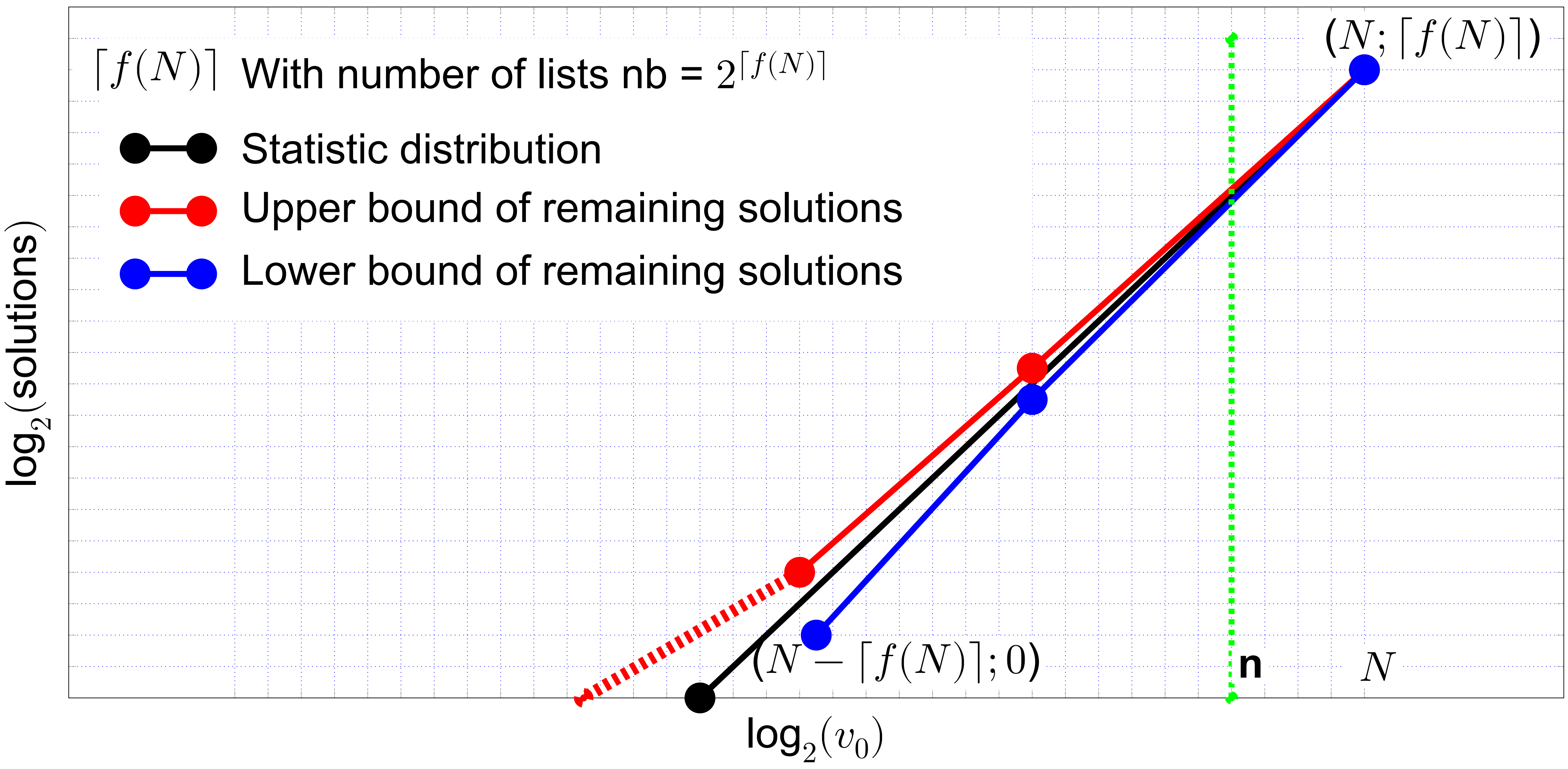
- $L_1(N\,, m_1\,, d_1) \leq L_2(N\,, m_2\,, d_2) \iff \forall\ 0 \leq n \leq N$, we have $m_{n,L_1} \leq m_{n,L_2}$, meaning that the number of "type 1" transitions in the sublist of length n of list L_1 is less than that of the sublist of length n of list L_2, and this holds for all sublists.
- $L_1(N\,, m_1\,, d_1) \geq L_2(N\,, m_2\,, d_2) \iff \forall\ 0 \leq n \leq N$, we have $m_{n,L_1} \geq m_{n,L_2}$, meaning that the number of "type 1" transitions in the sublist of length n of list L_1 is greater than that of the sublist of length n of list L_2, and this holds for all sublists.

Solutions of a list $L(N, m, d)$

Theorem:

- For each list $L(N, m, d)$, there exists a unique (minimal) solution $s_0 \in [0; 2^N[$ that follows this trajectory. With the convention 0 solution of $N\times$"0" instead of 2^N
- The infinite number of other solutions are of the form $s_0 + a \times 2^N$ with $a \geq 0$ since only the N least significant bits determine the first N transitions

Random List Theorem



Let a set of $nb = 2^{f(N)}$ transition lists of length N, independently and randomly generated. Each list $\mathcal{L}(N, m, d)$ may contain an arbitrary proportion m/N of type 1 transitions, without any specific constraint.

For a given integer $n < N$, let R_n denote the number of minimal initial values $v_0 < 2^n$ among the set of transition lists.

Then R_n follows the binomial distribution: $R_n \sim \text{Bin}\left(2^{f(N)}, \frac{1}{2^{N-n}}\right)$

This distribution follows directly from the independence of the lists and the successive filtering mechanism applied to the last $N - n$ transitions.

Define: $e := n - N + \lceil f(N) \rceil$

(i) Bounds via the Central Limit Theorem.

Let $4 \le z \le 6$ be a real number. Then, with probability at least $1 - \varepsilon$, where $\varepsilon = e^{-z^2/2}$:

- if $e \ge 7$, then $\quad R_n \ge 64 - 8\sqrt{2}z$,
- if $e \le 6$, then $\quad R_n \le 64 + 8z$.

(ii) Bounds via the Berry–Esseen inequality.

For any $\varepsilon < 10^{-3}$, define: $K := \left\lceil 2 \cdot \log_2\left(\frac{0.56}{\varepsilon}\right) \right\rceil + 1$

Then, with probability at least $1 - \varepsilon$, we have:

- if $e > K$, then $\quad R_n > \min := 2^{K-1} - \sqrt{2\ln(1/\varepsilon)} \cdot \sqrt{2^K}$,
- if $e < K$, then $\quad R_n < \max := 2^K + \sqrt{2\ln(1/\varepsilon)} \cdot \sqrt{2^K}$.

For $\varepsilon = 10^{-3}$, we have $K = 20$ and max $= 1{,}052{,}383$.

Heuristic approach: there is no solution if $e < -7$.

Set of lists Up(N, v_0)

Definition: Up(N, v_0) is the set of transition lists for which v_0 is the minimum of the orbit of the first N values or for all $0 < n \leq N, v_n \geq v_0$

Goal:

- Find the minimal list (and thus a lower bound of m)
- Find an upper bound on the number of lists
- Apply the "Random List Theorem" to prove that no solution exists (fragile zone)

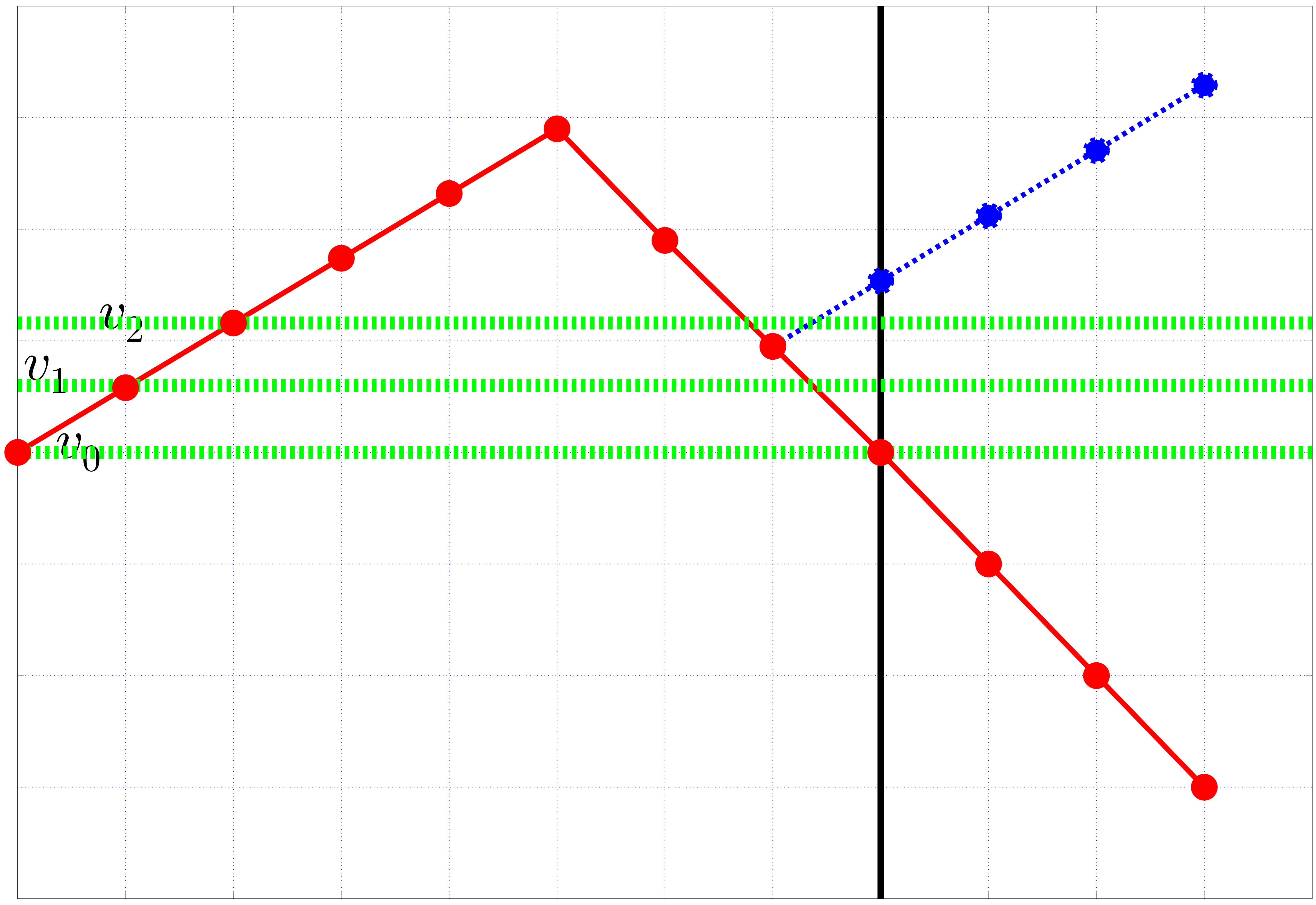
Set of lists Up'$_{2p}$(N, v_0)

Idea: Isolate at least $p > 30$ other values of the trajectory starting from v_0 to apply the "Random List Theorem" to obtain a contradiction in a proper zone.

Definition: Up'$_{2p}(N, v_0)$ is the set of transition lists for which, for all $0 < n \leq N, v_n \geq v_0/2^{2p}$

This allows filtering at least $p + 1$ values of the trajectory if for all $0 < n \leq N, v_n \geq v_0$ because for all $0 < p < N$, we have $v_{N+p}/v_p \geq 1/2^{2p} \times v_N/v_0$ and therefore $v_{N+p} \geq 1/2^{2p} v_p$

This comes from the fact that $v_1 < 2v_0$ and $v_{N+1} \geq v_N/2$



Approximated reduced Syracuse sequence : $v^{'}$

Let's define the approximated sequence by replacing the term $3v_n + 1$ by $3v_n$

It is immediate that the approximation makes sense for values of v_0 that are large enough and values of n that are low enough.

We ensure that the transitions from $v^{'}$ are identical to those of v

$$\begin{cases} v^{'}_0 = v_0 > 0 \\ v^{'}_{n+1} = \frac{v^{'}_n}{2} \quad \text{if} \quad v_n \quad \text{is} \quad \text{even (transition of type 0)} \\ v^{'}_{n+1} = \frac{3v^{'}_n}{2} \quad \text{if} \quad v_n \quad \text{is} \quad \text{odd (transition of type 1)} \end{cases}$$

Then $v_n = v^{'}_n + r_n$ and

$$\begin{cases} r_0 = 0 \\ r_{n+1} = r_n/2 \quad \text{if} \quad v_n \quad \text{is} \quad \text{even (transition of type 0)} \\ r_{n+1} = (3r_n + 1)/2 \quad \text{if} \quad v_n \quad \text{is} \quad \text{odd (transition of type 1)} \end{cases}$$

We have :

- $v_N = 3^m/2^N \times v_0 + r_N$: Of course, v_0 depends on $L(N, m, d)$ but we now have a factor $3^m/2^N$ that depends only on the global characteristics (N, m, d) of the transition list and a term r_N that does not depend on v_0 but only on the composition of the list $L(N, m, d)$. This is a first step towards using the "Random List Theorem".
- r_N is all the greater when the "type 0" transitions appear at the beginning of the list $L(N, m, d)$.
  r_N is maximum for the list L = d × "0" + m × "1" and equals $r_N = (3/2)^m - 1$, which is large.
  r_N is minimum for the list L = m × "1" + d × "0" and equals $r_N = 3^m/2^N - 1/2^d$, which can be very small.
- We have formulas to find r_N for $L = \sum_{i=1}^{k} p_i \times L_i + L_{k+1}$ where $p_i \in \mathbb{N}$

We focus on Up(N, v_0) hoping to have r_N sufficiently small.

Transition list Ceil(N)

Definition: Ceil(N) is the transition list such that for all $0 \leq n \leq N$, we have $m_n = \lceil ln(2) \times n/ln(3) \rceil$ where the function ln is the natural logarithm and m_n is the number of "type 1" transitions in the first n transitions.

We have:

- Ceil(N) $\in$ Up(N, v_0) because $r_n \geq 0$ and $m_n = \lceil ln(2) \times n/ln(3) \rceil \geq ln(2) \times n/ln(3)$, hence $v_n = v_n^{'} + r_n > v_n^{'} = 3^{m_n}/2^n \times v_0 \geq v_0$
- $r_N \approx 0.24048 \times 3^m/2^N \times m$ is of the order of magnitude of m by studying the patterns of Ceil(N).

Transition list JGL(N, v_0)

Definition: It is the minimal list of Up(N, v_0)

Property: JGL(N, v_0) = Ceil(N) as long as r_n does not offset an additional "type 0" transition (and then Ceil(N) maximizes r_N among the elements of Up(N, v_0) since the "type 0" transitions are placed at the head of the list).

To be able to add an additional "type 0" transition, it would therefore be necessary that:
$v_{N+1} = v_N/2 = 3^m/2^{N+1} + r_N/2 \geq v_0 \iff v_0 \leq (2^N \times r_N)/(2^{N+1} - 3^m)$ = Vmax(N)

The records of Vmax(N) are achieved for the fractions m/d, which are the best approximations of $X = ln(2)/(ln(3) - ln(2))$

Property: For $v_0 = 2^\alpha$ with $\alpha \geq 20$, we have JGL(N, v_0) = Ceil(N) for all $N \leq c_\alpha = \lceil 285\alpha \rceil$.

Therefore, there is no cycle of length strictly less than c_α, the conjecture having been verified up to 2^{20}.

Method 1

It can be shown that the *Random List Theorem* can be applied to the non-random set Up(N, v_0).

Here we use a heuristic remark which allows us to assume that there is no solution when $e < -7$.

We prove by induction the Syracuse conjecture, which is equivalent to: For any value $v_0 > 1$, there exists n such that $v_n < v_0$

The property holds for $v_0 < 2^{20}$

We show that a contradiction arises, that no solution $v_0 = 2^\alpha$ exists in Up(c_α, v_0).

The advantage of having studied JGL is to have the minimum of m, which is $m_{min} = \lceil ln(2) \times N/ln(3) \rceil$

We then very roughly upper bound the number of lists in Up(N, v_0) by $1/N \times \sum_{m=m_{min}}^{N} \binom{N}{m}$, as if there were no constraints on the transition lists, and we consider only the list that gives the minimum v_0 of the orbit among the N circular permutations.

Again, we make a very loose upper bound by bounding each term by the largest $\binom{N}{m_{min}}$ and use Stirling's formula for factorial approximations.

We then find the cardinality of Up(N, v_0) = nb = $2^{f(N)} < 2^{0.953N}$.

We apply the "heuristic remark" with $\alpha > 20$, $n = \alpha$, and $N = \lceil\ 285\alpha\ \rceil$ and look for a solution.

$e = n - N + f(N) < \alpha - 285\alpha + 0.953 \times 285\alpha = -12.395\alpha < -12.395 \times 20 < -247.9 < -7$ (e is a decreasing function of α)

We could therefore conclude that there is no solution and thus that induction property holds at rank v_0 but this method is mathematically unsatisfactory.

Consistency with the ”glides” records

We test the *Heuristic Approach to Establishing the Existence of Solutions* of the *Random List Theorem*.

Here, we consider the list of ”high-altitude flight” records maintained by Eric Roosendaal, which appear to be extreme outliers.

In fact, we observe that the order of magnitude of the g_k values is completely normal ($|e| < 6$ for each g_k), since on average, we have $|e| \approx \log_2(3)$—that is, g_k differs from the “central” value by only a factor of 3.

Here is the table of results obtained for the current 34 values of g_k, sorted by increasing value of e (the most spectacular value of g_k listed first due to its smallness)

k	$g_k = 2^n$	n	N_u	$N = N_v$	$Up(N) = 2^{f(N)}$	f(N)/N	e
30	$1008932249296231 \approx 2^{49.842}$	49.842	1445	886	$8.715456107851815 \times 10^{249}$	0.9371	-5.875
3	$27 \approx 2^{4.755}$	4.755	96	59	1108067472387578	0.8471	-4.268
23	$12235060455 \approx 2^{33.51}$	33.51	892	547	$2.0400397588223227 \times 10^{153}$	0.931	-4.206
32	$180352746940718527 \approx 2^{57.324}$	57.324	1575	966	$6.142175008565576 \times 10^{272}$	0.9381	-2.493
18	$63728127 \approx 2^{25.925}$	25.925	613	376	$4.4392258035308905 \times 10^{104}$	0.9246	-2.444
19	$217740015 \approx 2^{27.698}$	27.698	644	395	$1.1186587898698409 \times 10^{110}$	0.9255	-1.728
26	$13179928405231 \approx 2^{43.583}$	43.583	1122	688	$3.052651699851175 \times 10^{193}$	0.9342	-1.674
27	$31835572457967 \approx 2^{44.856}$	44.856	1161	712	$2.1622848174030833 \times 10^{200}$	0.9347	-1.646
4	$703 \approx 2^{9.457}$	9.457	132	81	$1.4459101842589664 \times 10^{21}$	0.8678	-1.25
28	$70665924117439 \approx 2^{46.006}$	46.006	1177	722	$1.5685756911708824 \times 10^{203}$	0.9349	-0.993
33	$1236472189813512351 \approx 2^{60.101}$	60.101	1614	990	$4.380651051074346 \times 10^{279}$	0.9383	-0.95
22	$2788008987 \approx 2^{31.377}$	31.377	729	447	$6.779198545151811 \times 10^{124}$	0.9277	-0.943
1	$3 \approx 2^{1.585}$	1.585	6	4	3	0.3962	-0.83
21	$1827397567 \approx 2^{30.767}$	30.767	706	433	$7.202752346027231 \times 10^{120}$	0.9272	-0.753
34	$2602714556700227743 \approx 2^{61.175}$	61.175	1639	1005	$7.88045853409778 \times 10^{283}$	0.9384	-0.741
2	$7 \approx 2^{2.807}$	2.807	11	7	13	0.5286	-0.492
25	$2081751768559 \approx 2^{40.921}$	40.921	988	606	$1.3353477308952514 \times 10^{170}$	0.9326	0.066
31	$118303688851791519 \approx 2^{56.715}$	56.715	1471	902	$3.243224444109984 \times 10^{254}$	0.9373	0.182
14	$13421671 \approx 2^{23.678}$	23.678	468	287	$2.235818694019155 \times 10^{79}$	0.9184	0.271
12	$1126015 \approx 2^{20.103}$	20.103	365	224	$3.0978023771135267 \times 10^{61}$	0.9119	0.372
20	$1200991791 \approx 2^{30.162}$	30.162	649	398	$7.870262801515273 \times 10^{110}$	0.9256	0.55
16	$26716671 \approx 2^{24.671}$	24.671	486	298	$2.9391335916232394 \times 10^{82}$	0.9193	0.625
15	$20638335 \approx 2^{24.299}$	24.299	476	292	$6.024579568492399 \times 10^{80}$	0.919	0.644
6	$35655 \approx 2^{15.122}$	15.122	220	135	$2.1289601394550887 \times 10^{36}$	0.8939	0.801
5	$10087 \approx 2^{13.3}$	13.3	171	105	$7.89573228439857 \times 10^{27}$	0.8826	0.973
17	$56924955 \approx 2^{25.763}$	25.763	502	308	$2.0971698034662947 \times 10^{85}$	0.9202	1.195
24	$898696369947 \approx 2^{39.709}$	39.709	897	550	$1.430402916945964 \times 10^{154}$	0.9311	1.802
9	$381727 \approx 2^{18.542}$	18.542	282	173	$1.1355686345484767 \times 10^{47}$	0.9035	1.856
13	$8088063 \approx 2^{22.947}$	22.947	401	246	$5.3011571416352496 \times 10^{67}$	0.9145	1.923
7	$270271 \approx 2^{18.044}$	18.044	267	164	$3.3605035857724012 \times 10^{44}$	0.9019	1.958
29	$739448869367967 \approx 2^{49.393}$	49.393	1187	728	$7.848211180044482 \times 10^{204}$	0.935	2.039
8	$362343 \approx 2^{18.467}$	18.467	269	165	$6.14186405300239 2 \times 10^{44}$	0.9017	2.251
10	$626331 \approx 2^{19.257}$	19.257	287	176	$7.828140161154488 \times 10^{47}$	0.904	2.356
11	$1027431 \approx 2^{19.971}$	19.971	298	183	$7.882388586673839 \times 10^{49}$	0.9058	2.724

Mean of |e| = 1.5845294117647057 $\approx \log_2(3)$

What seems chaotic in fact reveals an underlying continuum.

Transition list JGL$_{2p}(N, v_0)$

Definition valid only for $v_0 \in$ Up(N, v_0) :

- JGL$_{2p}(N, v_0)$ = $2p\times$"0" + JGL$(N-2p, v_0)$
- Up$_{2p}(N, v_0)$ is the set of lists greater than JGL$_{2p}(N, v_0)$

Properties:

- The first p values of the orbit of v_0 are within the interval $[v_0; 2^p v_0[$
- $m_{JGL_{2p}(N,v_0)} = \lceil ln(2) \times (N-2p)/ln(3)\rceil$ for $N \leq c_\alpha = \lceil 2{,}017{,}000 \times \alpha\rceil$ with $\alpha \geq 48$, and $v_0 = 2^\alpha$

Method 2

"Method 2" is identical to "Method 1", but:

- The set of lists is Up$_{2p}(N, v_0)$ whose cardinality is $n_0 = 2^{f(N)}$ with $f(N) < 1.2\,p + 0.95\,N$
- $m_{\text{min}} = \lceil \text{ln}(2) \times (N-2p)/\,\text{ln}(3)\rceil$
- We take $p = 2{,}000{,}000$, so we should find at least 2,000,001 elements of the orbit
- We apply the "Random List Theorem" in a zone that is not questionable

We apply the "Random List Theorem" with $\alpha > 48$ and $p = 2{,}000{,}000$, thus with $n = \alpha + p$ and N $= \lceil 2{,}017{,}000\alpha\rceil$, and we should find at least $p + 1 = 2{,}000{,}001$ solutions.
$e = n - \text{N} + f(\text{N}) < 2.2p - 100{,}849\alpha$ which is a decreasing function of α

So $e < -440{,}800 \ll 20$, therefore according to the Theorem, there are fewer than $1{,}052{,}383$ solutions.

We can thus conclude that there are fewer than $2{,}000{,}001$ solutions (since $2{,}000{,}001 > 1{,}052{,}383$), which gives a contradiction, and the entire inductive property is verified at rank v_0.

This time, the Theorem is used in a zone that is not questionable.

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