Answers to questions about the Syracuse sequence

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Abstract

The preprint `arXiv:2107.11160v4`, entitled *"Is the Syracuse Falling Time Bounded by 12?"*, authored by Shalom Eliahou, Jean Fromentin, and Rénald Simonetto, presents a collection of conjectures and questions concerning the *falling time* in the standard iteration of the Syracuse sequence (also known as the $3x + 1$ sequence).

In this note, we show that the set of conjectures formulated in that work can be addressed in a unified manner by applying the *Random List Theorem*.

This theorem provides a decisive tool for analyzing the dynamics of the Syracuse sequence. In particular, it allows one to give formal answers to questions concerning the falling time and provides a systematic method for studying such properties.

It is important to note that the empirical regularities motivating the conjectures originate from a statistical bias inherent in the observable data set, which corresponds to the range of integers less than 2^{68}. Beyond this bound, asymptotic behaviors emerge that contradict the initial regularities, even though the underlying dynamical rules remain unchanged.

In particular, we establish the theoretical existence of counterexamples to several of the conjectures proposed in `arXiv:2107.11160v4`. However, due to current computational limitations, it is not possible to exhibit such counterexamples explicitly.

1 Introduction

The study of the dynamical behavior of the Syracuse sequence (also known as the $3x+1$ iteration) continues to generate significant interest in mathematics. One recent approach focuses on analyzing the *falling time* of an integer n, which is defined as the number of iterations required for the sequence to produce a value less than n.

In this context, the document *SE* [3], authored by Shalom Eliahou, Jean Fromentin, and Rénald Simonetto, proposes a collection of precise conjectures based on numerical observations performed up to 2^{50}. These conjectures concern, in particular, the integers that achieve maximal falling times within specific intervals, as well as the structure of their trajectories under the iteration of the function T.

The present article provides answers to these conjectures by relying exclusively on the *Random List Theorem*. This theorem enables one to establish the systematic existence of integers whose behavior contradicts the regularities observed at small scales in the Syracuse sequence.

The method reduces the problem to an explicit enumeration of admissible transition lists for each case under consideration. In the present context, these enumerations are significantly more complex than those involved in the proof of the conjecture itself [1], due to the specific cases studied in [3].

In particular, we show that Conjectures **(5.1)** and **(5.2)**, as formulated in [3], are either incorrect or can be improved. Moreover, we provide detailed answers to the questions raised in **Section 3.2**, as well as those stated in the **title**, on **page 8**, on **page 9**, and in the **final challenge** of that document.

For each of these statements, we highlight the source of the empirical bias that led to their formulation. This bias results from the intrinsic limitations of the numerical data, which is restricted to integers less than 2^{68}. We show that, in the asymptotic regime, the observed tendencies reverse and the conjectures cannot be maintained.

The counterexamples predicted by our analysis are purely theoretical: their existence is proved, but no explicit instance can currently be exhibited due to computational limitations.

For the sake of a self-contained exposition, the results and theorems in Sections 2, 3, 4, 5, and 6 have been reproduced from the unpublished manuscript *"A Combinatorial Proof of the Syracuse Conjecture Using Transition Lists"* [1], authored by the present author.

The reader is also invited to consult the document *SE* [3], which the present article addresses in detail.

In Section 7, we address the questions and conjectures involving $\mathrm{sft}(n)$.

In Section 8, we consider those involving $\mathrm{ft}(n)$.

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This article constitutes a formalized synthesis of various communications by the author concerning the document [3] (denoted *SE* hereafter). These include the email sent to the authors on December 3, 2024, the presentation given on December 18, 2024, at the LMPA laboratory in Calais (France), and subsequent exchanges.

Finally, a complementary HTML+JavaScript file [2] allows the reader to reproduce all the computations presented both in the document *SE* and in the present article.

2 Definitions

2.1 Reduced Syracuse Sequence: (v_n)

$$v_{n+1} = \begin{cases} \dfrac{v_n}{2}, & \text{if } v_n \text{ is even (type 0)}, \\ \dfrac{3v_n+1}{2}, & \text{if } v_n \text{ is odd (type 1)}, \end{cases} \qquad \text{with } v_0 > 0.$$

Remarque 2.1. The parity of v_n still determines the type of transition. While the sequence could be written as $v_n = T^{(n)}(v_0)$, we retain the recurrence form.

2.2 Transition List $\mathcal{L}(N, m, d)$

A *transition list* of length N is a sequence of N transition types $t_i \in \{0, 1\}$, representing type 0 and type 1 transitions, respectively. It is denoted:

$$\mathcal{L}(N, m, d) = (t_0, t_1, \ldots, t_{N-1}),$$

where m is the number of type 1 transitions, and $d = N - m$ is the number of type 0 transitions.

- m: total number of type 1 transitions (*multiplications*);
- d: number of type 0 transitions (*divisions by 2*);
- $N = m + d$: total length of the transition list.

For each prefix of the list of length $n \leq N$, we define:

- $m_n = \sum_{i=0}^{n-1} \mathbb{1}_{\{t_i=1\}}$, the number of type 1 transitions among the first n elements;
- $d_n = n - m_n$, the number of type 0 transitions among the first n elements.

Exemple 2.2. For $v_0 = 7$, the sequence is:

$$7 \xrightarrow{1} 11 \xrightarrow{1} 17 \xrightarrow{1} 26 \xrightarrow{0} 13.$$

Then: $\mathcal{L}(4, 3, 1) = (1, 1, 1, 0)$.

Remarque 2.3. The list $\mathcal{L}(N, m, d)$ is also called a *parity vector*, since each t_i corresponds to the least significant bit of v_i.

2.3 Partial Order on Transition Lists

We define a partial order $\preccurlyeq$ on transition lists of length N by comparing the cumulative number of type 1 transitions at each prefix of the list.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two transition lists of length N. We write:

$$\mathcal{L}_1 \preccurlyeq \mathcal{L}_2 \quad \text{if and only if} \quad \text{for all } 0 \leq n \leq N, \quad m_{n,\mathcal{L}_1} \leq m_{n,\mathcal{L}_2},$$

where $m_{n,\mathcal{L}}$ denotes the number of type 1 transitions among the first n elements of list $\mathcal{L}$.

This relation is a partial order: it satisfies reflexivity, antisymmetry, and transitivity.

We also define the associated strict order:

$$\mathcal{L}_1 \prec \mathcal{L}_2 \quad \text{if and only if} \quad \text{for all } 0 \leq n \leq N, \quad m_{n,\mathcal{L}_1} < m_{n,\mathcal{L}_2}.$$

Remarque 2.4. This is not a total order. There may exist two lists $\mathcal{L}_1$ and $\mathcal{L}_2$ such that neither $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ nor $\mathcal{L}_2 \preccurlyeq \mathcal{L}_1$ holds. In such cases, the lists are said to be incomparable under this relation. This situation arises when the distribution of type 1 transitions differs in position but not in number.

Exemple 2.5. Let $\mathcal{L}_1 = (1, 0, 1)$ and $\mathcal{L}_2 = (0, 1, 1)$. The cumulative sums of type 1 transitions yield:

$$(m_1, m_2, m_3) = (1, 1, 2) \quad \text{for } \mathcal{L}_1, \quad \text{and} \quad (0, 1, 2) \quad \text{for } \mathcal{L}_2.$$

Thus, neither $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ nor $\mathcal{L}_2 \preccurlyeq \mathcal{L}_1$ holds: the lists are incomparable.

Remarque 2.6 (Interpretation)**.** This order reflects the temporal positioning of type 1 transitions: a list that accumulates multiplications more slowly (i.e., later in the sequence) is considered "smaller" in this ordering.

2.4 Solutions of a Transition List

We say that the initial or starting value v_0

- *follows* the transition list $\mathcal{L}_N$,
- *realizes* the transition list $\mathcal{L}_N$,
- or is a *solution* of the transition list $\mathcal{L}_N$,

if and only if the first N transitions of the reduced Syracuse sequence starting from v_0 are exactly those specified by $\mathcal{L}_N$.

We say that v_0 is the *minimal solution* of $\mathcal{L}_N$ if $v_0 < 2^N$. The existence of such a solution will be established in Section 3.4.

2.5 Approximate Reduced Syracuse Sequence: (v'_n)

We now introduce an approximate version of the reduced Syracuse sequence by neglecting the constant term in the type 1 transition. Specifically, in place of the expression $3v_n + 1$, we consider only $3v_n$. The resulting sequence (v'_n) is defined by the recurrence:

$$v'_{n+1} = \begin{cases} \dfrac{v'_n}{2}, & \text{if } v_n \text{ is even (type 0)}, \\ \dfrac{3v'_n}{2}, & \text{if } v_n \text{ is odd (type 1)}, \end{cases} \qquad \text{with } v'_0 = v_0.$$

Remarque 2.7. This approximation is especially meaningful when the initial value v_0 is large and the number of steps n remains moderate. Crucially, the transition types of the approximate sequence v' coincide exactly with those of the original sequence v, since the parity (and thus the transition vector (t_i)) is preserved.

However, the values of v'_n may be non-integer, which introduces a discrepancy compared to the actual sequence. To quantify this difference, we define a correction term r_n such that:

$$v_n = v'_n + r_n.$$

This decomposition will be used later to precisely analyze the divergence between the exact and approximate sequences.

3 Binomial Distribution of Initial Values Below a Threshold

Théorème 3.1 (Binomial Distribution of Minimal Initial Solutions)**.** *Let $nb \in \mathbb{N}$, and consider a set of nb independent and distinct transition lists $\mathcal{L}_1, \dots, \mathcal{L}_{nb}$, each of length N. Assume each list $\mathcal{L}(N, m, d)$ is random, with a proportion $p_{\mathcal{L}} = m/N$ of type 1 transitions. Let $k = N - n$ and R_k denote the number of minimal initial solutions $v_0 < 2^n = 2^{N-k}$ associated with these nb lists.*

Then, for any $0 < k < N - 10$, the random variable R_k follows the binomial distribution:

$$R_k \sim \mathrm{Bin}(nb, 1/2^k).$$

The proof of the theorem is broken down into several intermediate results, presented as lemmas in the following subsections.

3.1 Lemma: the Probability that v_1 is even is $\frac{1}{2}$ for $v_0 \geq 4$

Lemme 3.2. *Let $v_0 \geq 4$ be an integer chosen uniformly in the interval $[2^n, 2^{n+1})$ with $n \geq 2$. Then the parity of v_1, defined by the reduced Syracuse iteration*

$$v_1 = \begin{cases} v_0/2 & \textit{if } v_0 \equiv 0 \mod 2, \\ (3v_0+1)/2 & \textit{if } v_0 \equiv 1 \mod 2, \end{cases}$$

the parity of v_1 is uniformly distributed :

$$\mathbb{P}(v_1 \equiv 0 \mod 2) = \mathbb{P}(v_1 \equiv 1 \mod 2) = \frac{1}{2}.$$

Proof. Let us write the binary decomposition of v_0:

$$v_0 = \sum_{p=0}^{N} a_p \cdot 2^p, \quad \text{with } a_p \in \{0, 1\}.$$

Case 1: v_0 is even $(a_0 = 0)$

Then

$$v_1 = \frac{v_0}{2} = \sum_{p=1}^{N} a_p \cdot 2^{p-1} = \sum_{p=0}^{N-1} a_{p+1} \cdot 2^p.$$

The parity of v_1 is given by a_1. Since $N \geq 4$, the bit a_1 exists and is uniformly distributed in $\{0, 1\}$:

$$\mathbb{P}(v_1 \text{ even } \mid v_0 \text{ even}) = \mathbb{P}(a_1 = 0) = \frac{1}{2}.$$

Case 2: v_0 is odd ($a_0 = 1$)
We have:

$$v_1 = \frac{3v_0 + 1}{2} = \frac{1 + v_0 + 2v_0}{2}.$$

Replacing v_0 by its binary expansion:

$$v_1 = \frac{1 + \sum_{p=0}^{N} a_p \cdot 2^p + \sum_{p=0}^{N} a_p \cdot 2^{p+1}}{2} = \sum_{p=0}^{N+1} a'_p \cdot 2^p.$$

The least significant bit a'_0 depends on:

$$a'_0 = (1 + a_0 + a_1) \mod 2 = (1 + 1 + a_1) \mod 2 = a_1.$$

As in the even case, the parity of v_1 is determined by a_1, which is uniformly random. Hence:

$$\mathbb{P}(v_1 \text{ even } \mid v_0 \text{ odd}) = \mathbb{P}(a_1 = 0) = \frac{1}{2}.$$

□

Remarque 3.3. This lemma shows that the parity of v_1 is exactly balanced as soon as $v_0 \geq 4$, i.e., when the binary representation of v_0 has at least two digits. This is an exact property, not an asymptotic estimate.

Some sources incorrectly state that this equiprobability only holds "in sufficiently large intervals." For instance, the May 2025 version of the French Wikipedia article on the Syracuse conjecture claims:

> *"the parity of the result is independent of that of v, if v is randomly chosen in a sufficiently large interval."*

However, as the above proof shows, the property already holds perfectly for all $v_0 \geq 4$, without any asymptotic assumption.

It is also important to note that this equiprobability cannot be extended to subsequent values v_n, since the trajectory is deterministically correlated with v_0. Assuming independence along the entire sequence is a common error in probabilistic models of the Syracuse dynamics. While the lemma justifies local randomness at the first step, caution is required when extending this reasoning to full orbits.

3.2 Lemma: Bijection between Transition Lists of Length N and Minimal Initial Values $v_0 < 2^N$ That Realize Them

Lemme 3.4. *For every integer $N \geq 1$, there is a bijection between:*

- *the set $\mathcal{L}_N$ of binary transition lists $(t_0, \ldots, t_{N-1}) \in \{0, 1\}^N$;*
- *and the set of initial values $v_0 < 2^N$ so that the sequence $(v_1, \ldots, v_N)$ generated by the reduced Syracuse iteration follows the transition pattern $(t_0, \ldots, t_{N-1})$.*

Each transition list uniquely determines a minimal initial value $v_0 < 2^N$ that realizes it. Furthermore, all other values generating the same transition list are of the form:

$$v_0^{(n)} = v_0 + n \cdot 2^N, \quad n \in \mathbb{N}.$$

Remarque 3.5. This relies on extending the definition to include $v_0 = 0$, which is then considered as the minimal solution for all transition lists containing exactly N transitions of type 0 (and no transitions of type 1), instead of assigning $v_0 = 2^N$.

Proof. The reduced Syracuse dynamics assigns to any integer v_0 a transition list $(t_0, \ldots, t_{N-1})$ defined by:

$$t_i = \begin{cases} 0 & \text{if } v_i \text{ is even}, \\ 1 & \text{if } v_i \text{ is odd}, \end{cases}$$

where $v_{i+1} = T(v_i)$ with T the reduced Syracuse function.

We prove by induction on N that for each binary word of length N, there exists a unique minimal $v_0 < 2^N$ realizing it.

Base case $N = 1$ There are two possible transition lists:

- $t_0 = 0$ (even), realized by $v_0 = 0$ (with the extension);
- $t_0 = 1$ (odd), realized by $v_0 = 1$.

Each transition bit is thus realized by a unique $v_0 < 2$.

Inductive step Assume the result holds for lists of length N: for every $\mathcal{L}_N = (t_0, \ldots, t_{N-1})$, there exists a unique minimal value $s_0 < 2^N$ realizing it.

Let $\mathcal{L}_{N+1} = (t_0, \ldots, t_N)$ be a list of length $N + 1$.

By the inductive hypothesis, the prefix $(t_0, \ldots, t_{N-1})$ corresponds to a unique value $s_0 < 2^N$. Consider the two candidate initial values:

$$v_0^{(0)} = s_0, \quad v_0^{(1)} = s_0 + 2^N.$$

Both share the same lower N bits and thus follow the same first N transitions. Let m be the number of type 1 transitions among $(t_0, \ldots, t_{N-1})$. Then, by recurrence[1], their corresponding values at time N differ by 3^m:

$$v_N^{(a)} = s_N + a \cdot 3^m.$$

We now determine which of the two values $v_0^{(a)}$ satisfies t_N, by testing the parity of $v_N^{(a)}$:

- If $s_N \equiv t_N \pmod 2$, choose $a = 0$;
- Otherwise, choose $a = 1$.

Thus, exactly one of the two values $v_0^{(0)}$ or $v_0^{(1)}$ matches the full transition list $\mathcal{L}_{N+1}$, and its value is strictly less than 2^{N+1}.

Infinitely many solutions Since adding 2^N does not affect the first N transitions, any integer of the form:

$$v_0^{(n)} = v_0 + n \cdot 2^N, \quad n \in \mathbb{N},$$

also realizes the same transition list. Therefore, for each $\mathcal{L}_N$, there exists an infinite arithmetic progression of initial values with a unique minimal representative in $[0, 2^N)$.

□

Remarque 3.6 (On the precedence of the lemma)**.** In the standard case, this lemma corresponds to results previously established by Riho Terras (1976) [5] and C. J. Everett (1977) [4], as kindly pointed out to me by Shalom Eliahou in a personal correspondence dated December 18, 2024.

These references were not identified in earlier versions of this document (prior to version 3.1.2), as the original articles are written in English and adopt a different formalism.

That said, the main contribution of this section lies in the corollary that follows, which, to the best of our knowledge, constitutes a new result within the specific framework developed here.

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[1] Let us detail the first transition:
The value $v_0^{(a)}$ has the same parity as s_0, corresponding to $t_0 \in \{0, 1\}$.

- If $t_0 = 0$, then s_0 is even (since s_0 follows $\mathcal{L}_N$), and

$$v_1^{(a)} = \frac{v_0^{(a)}}{2} = \frac{s_0 + a \cdot 2^N}{2} = \frac{s_0}{2} + a \cdot 2^{N-1} = s_1 + a \cdot 2^{N-1}.$$

- If $t_0 = 1$, then s_0 is odd (since s_0 follows $\mathcal{L}_N$), and

$$v_1^{(a)} = \frac{3v_0^{(a)} + 1}{2} = \frac{3(s_0 + a \cdot 2^N) + 1}{2} = \frac{3s_0 + 1}{2} + \frac{a \cdot 3 \cdot 2^{N-1}}{2} = s_1 + a \cdot 3 \cdot 2^{N-1}.$$

The value $v_1^{(a)}$ has the same parity as s_1, which corresponds to t_1.

One can easily prove by induction that, for all $0 \le n \le N$,

$$v_n^{(a)} = s_n + a \cdot 3^{m_n} \cdot 2^{N-n},$$

where m_n denotes the number of type 1 transitions among the first n transitions of $\mathcal{L}_N$.
and for $n = N$:

$$v_N^{(a)} = s_N + a \cdot 3^m.$$

3.3 Corollary of Lemma 3.4: $\mathbb{P}(v_0 < 2^{N-1}) = \frac{1}{2}$ for Transition Lists of Length N

Corollaire 3.7. *Let $N \geq 1$. Among all transition lists of length N, the probability that the minimal initial value v_0 satisfies $v_0 < 2^{N-1}$ is exactly*

$$\mathbb{P}(v_0 < 2^{N-1}) = \frac{1}{2}.$$

Proof. We consider only the minimal initial values $v_0 < 2^N$ arising from the bijection of Lemma 3.4.

Given a list of length N, the construction extends a prefix of length $N-1$ by one final bit t_{N-1}. The two candidates for v_0 are:

$$v_0^{(0)} = s_0, \quad v_0^{(1)} = s_0 + 2^{N-1}.$$

Only one of these two values satisfies the final transition, depending on the parity of s_{N-1} and the bit t_{N-1}. The minimal representative $v_0 = s_0$ is selected if and only if:

$$(t_{N-1} = 0 \text{ and } s_{N-1} \text{ is even}) \quad \text{or} \quad (t_{N-1} = 1 \text{ and } s_{N-1} \text{ is odd}).$$

Assuming, as shown in Lemma 3.2, that $\mathbb{P}(s_{N-1} \text{ even}) = \frac{1}{2}$, and letting p denote the probability that $t_{N-1} = 1$, we compute:

$$\mathbb{P}(v_0 = s_0) = (1-p) \cdot \frac{1}{2} + p \cdot \frac{1}{2} = \frac{1}{2}.$$

Hence, among all transition lists of length N, the minimal initial value v_0 falls below 2^{N-1} with probability exactly

$$\mathbb{P}(v_0 < 2^{N-1}) = \frac{1}{2}.$$

□

3.4 Corollary of Lemma 3.4: $\mathbb{P}(v_0 < 2^{N-k}) = \frac{1}{2^k}$ for Transition Lists of Length N

Corollaire 3.8. *Let $N \geq 1$ and $0 \leq k \leq N$. Among all transition lists of length N, the probability that the associated minimal initial value satisfies $v_0 < 2^{N-k}$ is exactly*

$$\mathbb{P}(v_0 < 2^{N-k}) = \frac{1}{2^k}.$$

Proof. By iterating the reasoning of Corollary 3.7 k times, we observe that each additional transition bit splits the space of minimal initial values in half. Starting from the full interval $[0, 2^N)$, the probability that a randomly constructed list yields a minimal v_0 below 2^{N-k} is thus

$$\mathbb{P}(v_0 < 2^{N-k}) = \frac{1}{2^k}.$$

This also yields the following consequences:

- The probability that v_0 falls in the interval $[2^{N-k}, 2^{N-k+1})$ is likewise $\frac{1}{2^k}$;
- By complement, the probability that $v_0 \geq 2^{N-k}$ is $1 - \frac{1}{2^k}$.

Finally, to have a nonzero expected number of minimal values $v_0 < 2^{N-k}$ in a sample of $n_0 = 2^{f(N)}$ transition lists, we require:

$$\frac{2^{f(N)}}{2^k} > 1 \quad \text{if and only if} \quad k < f(N).$$

This inequality gives a critical threshold beyond which the probability of sampling such a value becomes negligible. □

Remarque 3.9. This exact power distribution is crucial in establishing bounds that scale logarithmically with N in the Random List Theorem. It reflects the uniform binary structure induced by the bijection of Lemma 3.4.

3.5 Iterated Binomial Reduction

Lemme 3.10 (Iterated Binomial Reduction)**.** *Let $nb \in \mathbb{N}$, and define a sequence of random variables $(R_k)_{k \geq 0}$ recursively by:*

$$R_0 = nb, \quad \text{and} \quad R_k \sim \mathrm{Bin}(R_{k-1}, 1/2) \quad \text{for all } k \geq 1.$$

Then, for every $k \in \mathbb{N}$, the random variable R_k follows the binomial distribution:

$$R_k \sim \mathrm{Bin}\left(nb, \frac{1}{2^k}\right).$$

Proof. We proceed by induction on k.

Base case: for $k = 0$, we have $R_0 = nb$, which is equivalent to $R_0 \sim \text{Bin}(nb, 1)$, i.e., $R_0 \sim \text{Bin}(nb, 1/2^0)$.

Inductive step: suppose that for some $k \geq 0$, we have

$$R_k \sim \text{Bin}\left(nb, \frac{1}{2^k}\right).$$

Then, conditionally on $R_k = r$, the next variable satisfies

$$R_{k+1} \mid R_k = r \sim \text{Bin}(r, 1/2).$$

Thus, we can write:

$$R_{k+1} = \sum_{i=1}^{R_k} Y_i,$$

where the Y_i are independent Bernoulli(1/2) variables, independent of R_k.

Since $R_k \sim \text{Bin}(nb, 1/2^k)$, we can express:

$$R_k = \sum_{i=1}^{nb} X_i, \quad \text{where } X_i \sim \text{Bernoulli}(1/2^k),$$

and the X_i are independent.

Each $X_i = 1$ indicates that the i-th item survived the first k filtering steps. For R_{k+1}, we apply one more independent Bernoulli(1/2) filtering to each $X_i = 1$.

Therefore, each $i \in \{1, \dots, nb\}$ survives the first $k+1$ steps with probability:

$$\mathbb{P}(\text{survival}) = \frac{1}{2^k} \cdot \frac{1}{2} = \frac{1}{2^{k+1}}.$$

By independence, we conclude that:

$$R_{k+1} \sim \text{Bin}\left(nb, \frac{1}{2^{k+1}}\right).$$

Conclusion: the result follows by induction: for all $k \in \mathbb{N}$,

$$\boxed{R_k \sim \text{Bin}\left(nb, \frac{1}{2^k}\right).}$$

□

3.6 Proof ot the Theorem

Proof. Each transition list $\mathcal{L}$ defines a unique minimal solution $v_0 < 2^N$ under the convention that $v_0 = 0$ corresponds to the all-zero transition list (see Lemma 3.4).

For each transition t_{N-k-1} in each list, we consider v_0 to be the minimal initial value that solves the first $N - k - 1$ transitions.

We know that $v_0 < 2^{N-k-1}$.

Moreover, v_0 is also the minimal solution for the first $N-k$ transitions of $\mathcal{L}$ if and only if t_{N-k-1} matches the "natural" transition from v_0, that is, if

$$((t_{N-k-1} = 1) \text{ and } \; v_{N-k-1} \text{ is odd}) \quad \text{or} \quad ((t_{N-k-1} = 0) \text{ and } \; v_{N-k-1} \text{ is even}).$$

The probability of this event is

$$p_{\mathcal{L}} \cdot \frac{1}{2} + (1 - p_{\mathcal{L}}) \cdot \frac{1}{2} = \frac{1}{2}.$$

Indeed, if the transition does not match, then the minimal solution for the first $N - k$ transitions of $\mathcal{L}$ would be $v_0 + 2^{N-k} \geq 2^{N-k}$, and thus no longer strictly below the threshold.

We now prove by induction that $R_k \sim \text{Bin}\left(nb, \frac{1}{2^k}\right)$.

Base case: For $k = 1$, we consider the final transition t_{N-1} of each transition list. Given that the minimal initial value v_0 for the first $N - 1$ transitions satisfies $v_0 < 2^{N-1}$, the value v_0 also solves the full list of N transitions if and only if t_{N-1} matches the natural parity transition induced by v_{N-1}. This occurs with probability $1/2$, since the transition is chosen at random and independently of v_0, and the parity of v_{N-1} is balanced in expectation.

Since the nb transition lists are all distinct and independent, we perform nb independent Bernoulli trials with success probability $1/2$, one for each list. It follows that

$$R_1 \sim \text{Bin}(nb, 1/2).$$

Inductive step: Assume that $R_{k-1} \sim \text{Bin}(nb, 1/2^{k-1})$.

By iterating the same reasoning at step k, after analyzing the last $k-1$ transitions of each list, each remaining minimal value survives the next transition with probability $1/2$, independently. Therefore,

$$R_k \sim \text{Bin}\left(R_{k-1}, \frac{1}{2}\right).$$

Then, by applying Lemma 3.10, we deduce that

$$R_k \sim \text{Bin}\left(nb, \frac{1}{2^k}\right).$$

This completes the proof by induction.

Therefore, we conclude that the number of minimal initial solutions strictly less than 2^{N-k} follows the binomial distribution $\text{Bin}(nb, 1/2^k)$.

□

4 Random List Theorem

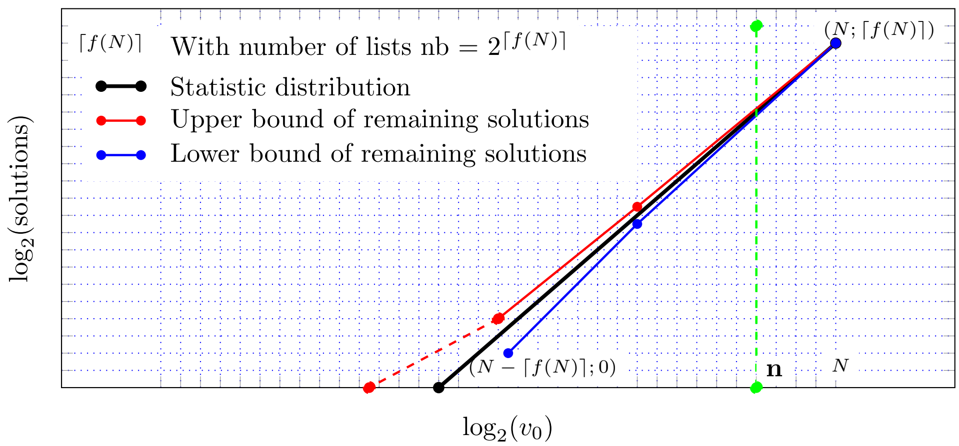

Figure 1: Number of solutions $v_0 < 2^n$

Remarque 4.1 (Idea)**.** The probability that the minimal initial value v_0 of a transition list $\mathcal{L}(N, m, d)$ satisfies $v_0 < 2^n$ is 2^{N-n}.

If $2^{f(N)}$ random lists are tested, then we expect

$$\mathbb{E}\left[\#\{v_0 < 2^n\}\right] \approx 2^{f(N)} \cdot 2^{n-N} = 2^e.$$

Hence, the shift index e provides a direct estimate of the expected number of solutions.

Théorème 4.2 (Random List Theorem)**.** *Let a set of $nb = 2^{f(N)}$ transition lists of length N, independently and randomly generated. Each list $\mathcal{L}(N, m, d)$ may contain an arbitrary proportion m/N of type 1 transitions, without any specific constraint.*

For a given integer $n < N$, let R_n denote the number of minimal initial values $v_0 < 2^n$ among the set of transition lists.

Then R_n follows the binomial distribution:

$$R_n \sim \text{Bin}\left(2^{f(N)}, \frac{1}{2^{N-n}}\right).$$

This distribution follows directly from the independence of the lists and the successive filtering mechanism applied to the last $N-n$ transitions.

Define:

$$e := n - N + \lceil f(N) \rceil.$$

(i) Bounds via the Central Limit Theorem.

Let $4 \leq z \leq 6$ be a real number. Then, with probability at least $1 - \varepsilon$, where $\varepsilon = e^{-z^2/2}$:

- *if $e \geq 7$, then* $\quad R_n \geq 64 - 8\sqrt{2}z$,
- *if $e \leq 6$, then* $\quad R_n \leq 64 + 8z$.

(ii) ***Bounds via the Berry–Esseen inequality.***
For any $\varepsilon < 10^{-3}$, define:

$$K := \left\lceil 2 \cdot \log_2 \left(\frac{0.56}{\varepsilon} \right) \right\rceil + 1.$$

Then, with probability at least $1 - \varepsilon$, we have:

- *if $e > K$, then* $R_n > \mathit{min} := 2^{K-1} - \sqrt{2\ln(1/\varepsilon)} \cdot \sqrt{2^K}$,
- *if $e < K$, then* $R_n < \mathit{max} := 2^K + \sqrt{2\ln(1/\varepsilon)} \cdot \sqrt{2^K}$.

The following values are guaranteed for some standard thresholds:

ε	K	*min*	*max*
10^{-3}	20	520,481	1,052,383
10^{-4}	26	33,519,272	67,144,024
10^{-5}	33	4,294,522,559	8,590,379,329

Proof. According to Theorem 3.1, we have

$$R_n \sim \mathrm{Bin}(nb, 1/2^{N-n}).$$

(i) Central Limit Theorem approximation:
Let $k = N - n$, the number of suffix transitions under analysis.
We apply the classical Central Limit Theorem to the sum of nb independent and identically distributed Bernoulli variables with constant parameter $p = 1/2^k$.
This sum defines the variable R_n, with expected value and standard deviation given by:

$$\mu := \mathbb{E}[R_n] = nb \cdot p = \frac{nb}{2^k},$$

$$\sigma := \sqrt{\mathrm{Var}(R_n)} = \sqrt{nb \cdot p(1-p)} = \sqrt{\frac{nb}{2^k}\left(1 - \frac{1}{2^k}\right)}.$$

As soon as $\mu = \frac{nb}{2^k} \gtrsim 30$, the normal approximation becomes accurate in practice. Asymptotically, we have convergence in distribution:

$$Z_n := \frac{R_n - \mu}{\sigma} \xrightarrow[nb\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

We now derive probabilistic bounds for R_n using a Gaussian tail threshold $z > 0$.

- **Upper bound (tail on the right):**

$$\mathbb{P}(Z_n < z) > 1 - \varepsilon \quad \text{whenever} \quad R_n < \mu + z \cdot \sigma, \quad \text{with } \varepsilon := 1 - \Phi(z).$$

We bound successively:

$$R_n < \frac{nb}{2^k} + z \cdot \sqrt{\frac{nb}{2^k}\left(1 - \frac{1}{2^k}\right)} < \frac{nb}{2^k} + z \cdot \sqrt{\frac{nb}{2^k}}.$$

Now suppose $nb = 2^{f(N)} \le 2^{\lceil f(N) \rceil}$. Then,

$$R_n < \frac{2^{\lceil f(N) \rceil}}{2^k} + z \cdot \sqrt{\frac{2^{\lceil f(N) \rceil}}{2^k}}.$$

Define $e := n - N + \lceil f(N) \rceil$. Then $e \le 6$ is equivalent to $k \ge \lceil f(N) \rceil - 6$. Since R_n is decreasing in k, the upper bound is maximal when $k = \lceil f(N) \rceil - 6$. Therefore:

$$\text{if } e \le 6 \text{ then } R_n < 64 + 8z.$$

- **Lower bound (tail on the left):**
Using the Central Limit Theorem, for any $z > 0$, we have:

$$\mathbb{P}(Z_n > z) > 1 - \varepsilon \quad \text{whenever} \quad R_n > \mu - z \cdot \sigma, \quad \text{with } \varepsilon := 1 - \Phi(z).$$

We start from the inequality:

$$R_n > \frac{nb}{2^k} - z \cdot \sqrt{\frac{nb}{2^k}\left(1 - \frac{1}{2^k}\right)} > \frac{nb}{2^{k-1}} - z \cdot \sqrt{\frac{nb}{2^{k-1}}}.$$

Now suppose $nb = 2^{f(N)} \geq 2^{\lceil f(N) \rceil - 1}$. Then:

$$R_n > \frac{2^{\lceil f(N) \rceil - 1}}{2^k} - z \cdot \sqrt{\frac{2^{\lceil f(N) \rceil}}{2^k}}.$$

Define $e := n - N + \lceil f(N) \rceil$. Then $e \geq 7$ is equivalent to $k \leq \lceil f(N) \rceil - 7$.
Since R_n is decreasing in k, the lower bound is minimal when $k = \lceil f(N) \rceil - 7$. Therefore:

$$\text{if } e \geq 7 \text{ then } R_n > 64 - 8\sqrt{2}z.$$

Numerical remark: For $z \geq 4$, the Mills ratio gives $z \approx \sqrt{2\ln(1/\varepsilon)}$, hence $\varepsilon \approx e^{-z^2/2}$.

(ii) Approximation with Berry–Esseen Inequality:

- **Berry–Esseen Inequality**
  We apply the Berry–Esseen inequality to the centered and normalized variable

$$Z_n := \frac{R_n - nb \cdot p}{\sqrt{nb \cdot p(1-p)}},$$

  where R_n denotes the number of minimal initial values below 2^n among a large set of $nb = 2^{f(N)}$ transition lists of length N. Although the process is fundamentally deterministic, the distribution of R_n can be approximated by that of a binomial variable $\text{Bin}(nb, p)$, with $p = 1/2^{N-n}$, based on probabilistic modeling of parity transitions.
  This allows us to apply the standard form of the Berry–Esseen inequality, which quantifies the convergence to the normal distribution for sums of independent and identically distributed Bernoulli(p) variables.
  The third absolute centered moment of a Bernoulli variable is given by

$$\rho = \mathbb{E}[|X-p|^3] = p(1-p)^3 + (1-p)p^3 = p(1-p)(1-2p+2p^2),$$

  which is finite for any fixed $p \in (0,1)$. The variance is $\sigma^2 = p(1-p)$, and the Berry–Esseen inequality yields:

$$|\mathbb{P}(Z_n \leq z) - \Phi(z)| \leq \frac{C \cdot \rho}{\sigma^3 \sqrt{nb}} = \frac{C \cdot (1-2p+2p^2)}{(p(1-p))^{1/2} \cdot \sqrt{nb}} = \frac{C_p}{\sqrt{nb}},$$

  with $C \leq 0.56$ an absolute constant.
  Let

$$C_p := \frac{C \cdot (1-2p+2p^2)}{(p(1-p))^{1/2}},$$

  which depends only on p. This formulation enables us to derive explicit quantitative bounds for the probability that R_n deviates from its expectation, using Gaussian approximations with computable error margins.

- **Getting the threshold**
  We aim to ensure that $\mathbb{P}(Z_k < z) > 1 - \varepsilon$, and we seek to determine for which values of nb this inequality holds.
  Approximating the Gaussian tail for large z using the classical Mills ratio :

$$1 - \Phi(z) \approx \frac{1}{z\sqrt{2\pi}} e^{-z^2/2},$$

  we substitute $z := \sqrt{2\ln(1/\varepsilon)}$, which yields:

$$1 - \Phi(z) \approx \frac{\varepsilon}{\sqrt{4\pi \ln(1/\varepsilon)}}.$$

  According to the Berry–Esseen inequality:

$$\mathbb{P}(Z_k < z) \geq \Phi(z) - \frac{C_p}{\sqrt{nb}}.$$

  Therefore, we require:

$$\Phi(z) - \frac{C_p}{\sqrt{nb}} > 1 - \varepsilon.$$

  By substituting the approximation for $\Phi(z)$, we obtain:

$$\frac{\varepsilon}{\sqrt{4\pi \ln(1/\varepsilon)}} + \frac{C_p}{\sqrt{nb}} < \varepsilon.$$

To simplify, note that for small ε, we have $\ln(1/\varepsilon) \gg 1$, so $\frac{\varepsilon}{\sqrt{4\pi \ln(1/\varepsilon)}} \ll \varepsilon$. Therefore, this term becomes negligible, and we may approximate the condition by:

$$\frac{C_p}{\sqrt{nb}} < \varepsilon, \quad \text{which implies} \quad nb > \left(\frac{C_p}{\varepsilon}\right)^2.$$

For large $N-n$ (i.e., when we filter over a large number of final transitions), we have $p = 1/2^{N-n} \ll 1$, and the constant becomes:

$$C_p = \frac{C \cdot (1 - 2p + 2p^2)}{\sqrt{p(1-p)}} \approx \frac{C}{\sqrt{p}}.$$

Substituting this into the bound yields the condition:

$$\boxed{nb > \left(\frac{C}{\varepsilon}\right)^2 \cdot 2^{N-n}.}$$

Taking logarithms (base 2), we obtain:

$$\log_2(nb) > 2\log_2\left(\frac{C}{\varepsilon}\right) + (N-n).$$

Let us define $nb = 2^{f(N)}$. Then the inequality becomes:

$$f(N) > 2\log_2\left(\frac{C}{\varepsilon}\right) + (N-n).$$

This is satisfied as soon as

$$\lceil f(N) \rceil - 1 \geq \left\lceil 2\log_2\left(\frac{C}{\varepsilon}\right) \right\rceil + (N-n).$$

Let us define the threshold:

$$K := \left\lceil 2\log_2\left(\frac{C}{\varepsilon}\right) \right\rceil + 1, \quad \text{and let} \quad e := n - N + \lceil f(N) \rceil.$$

Then the condition becomes simply:

$$e \geq K.$$

– **Upper bound (tail on the right):** By applying the Berry–Esseen inequality at depth $n_K = N - \lceil f(N) \rceil + K$ (i.e., when $e = K$), we obtain:

$$\mathbb{P}(Z_{n_K} \leq z) \geq 1 - \varepsilon, \quad \text{with } z = \sqrt{2\ln(1/\varepsilon)}.$$

Since $Z_{n_K} = \dfrac{R_{n_K} + \mu}{\sigma}$, this implies:

$$R_{n_K} < \mu + z \cdot \sigma, \quad \text{with probability at least } 1 - \varepsilon,$$

where

$$\mu = \mathbb{E}[R_{n_K}] = \frac{nb}{2^{N-n_K}} = \frac{2^{f(N)}}{2^{\lceil f(N) \rceil - K}} \leq 2^K,$$

and

$$\sigma = \sqrt{\mathrm{Var}(R_{n_K})} = \sqrt{\mu\left(1 - \frac{1}{2^{N-n_K}}\right)} < \sqrt{\mu} \leq \sqrt{2^K}.$$

Therefore, with probability at least $1 - \varepsilon$, we have:

$$R_{n_K} < 2^K + z \cdot \sqrt{2^K}.$$

Finally, since $R_n \leq R_{n_K}$ for all $e \leq K$ i.e. $n \leq n_K$ (as the sequence R_k is increasing in k), the upper bound on R_{n_K} also applies to R_n.

$$\text{if } e \leq K \text{ then } R_n < 2^K + z \cdot \sqrt{2^K}.$$

– **Lower bound (tail on the left):**
Since
$$|\mathbb{P}(Z_n \le z) - \Phi(z)| = |\mathbb{P}(Z_n \ge -z) - \Phi(-z)|,$$
we may reuse the previous estimates in the opposite tail.
By applying the Berry–Esseen inequality at depth $n_K = N - \lceil f(N) \rceil + K$ (i.e., when $e = K$), we obtain:
$$\mathbb{P}(Z_{n_K} \ge -z) \ge 1 - \varepsilon, \quad \text{with } z = \sqrt{2\ln(1/\varepsilon)}.$$
Since $Z_{n_K} = \dfrac{R_{n_K} - \mu}{\sigma}$, this implies:
$$R_{n_K} > \mu - z \cdot \sigma, \quad \text{with probability at least } 1 - \varepsilon,$$
where
$$\mu = \mathbb{E}[R_{n_K}] = \frac{nb}{2^{N-n_K}} = \frac{2^{f(N)}}{2^{\lceil f(N) \rceil - K}} \ge 2^{K-1},$$
and
$$\sigma = \sqrt{\mathrm{Var}(R_{n_K})} = \sqrt{\mu\left(1 - \frac{1}{2^{N-n_K}}\right)} < \sqrt{\mu} \le \sqrt{2^K}.$$
Therefore, with probability at least $1 - \varepsilon$, we have:
$$R_{n_K} > 2^{K-1} - z \cdot \sqrt{2^K}.$$
Finally, since $R_n \ge R_{n_K}$ for all $e \ge K$ i.e. $n \ge n_K$ (as the sequence R_k is increasing in k), the lower bound on R_{n_K} also applies to R_n.
$$\text{if } e \ge K \text{ then } R_n > 2^{K-1} - z \cdot \sqrt{2^K}.$$

□

Remarque 4.3 (Random List Theorem for Non-Random Sets of Transition Lists)**.**

Conclusion. For sets of transition lists delimited by suitable boundaries, the *Random List Theorem* can be applied without any special modification.

In the proofs, we would like to apply the *Random List Theorem* to sets of transition lists that are neither random nor independent.

If one were to apply the theorem to the entire set of 2^N transition lists of length N, then for every $0 < n \le N$ we would obtain $R_n = 2^{N-n}$ by the bijection (see 3.4), and nothing would be random. The difficulty is that if one considers an arbitrary subset of transition lists, without any specific structural property, the extreme cases cannot be excluded, which makes it difficult to draw any meaningful conclusion.

To overcome this difficulty, recall that m_n denotes the number of type 1 transitions among the first n transitions of $\mathcal{L}$. With this notation in place, we shall apply the *Random List Theorem* to a family of transition lists $\mathcal{L}(N, m, d)$ satisfying the condition
$$m_n \ge \lceil kn \rceil \quad \text{for all } 0 < n \le N,$$
together with either $\lceil kN \rceil \le m \le N$ or $m = \lceil kN \rceil$, where $k = \ln(2)/\ln(3)$, for instance for the list $\mathrm{Ceil}(N)$ that we shall study later in Section 6.

Each such list, as in Figure 2, can be interpreted as a discrete path from $(0, 0)$ to (d, m) consisting of N elementary steps, where each step is either:

- a horizontal move (type 0 transition), increasing d by 1; or
- a vertical move (type 1 transition), increasing m by 1.

In the diagram above:

- The blue line represents the Ceil boundary (a constraint to be respected);
- The green line is a valid transition list, always staying above Ceil;
- The black line is the classical boundary of the Catalan triangle (without the Ceil constraint);
- Green points indicate the endpoints of valid transition lists for $N = 15$ (only the intersection point with $\mathrm{Ceil}(N)$ when we restrict to $m = k \cdot N$);

Note that transition lists passing through the points on the vertical axis $(0, n)$ have the minimal solution $v_0 \ge 2^{n-1}$.

The number of lists passing through each point (d, m) is at least on the order of N, which is very large, except at $(0, m)$ where m is the extremal value of m; in that case, there is only a single list, but its minimal solution is far too large and does not belong to the set of admissible solutions.

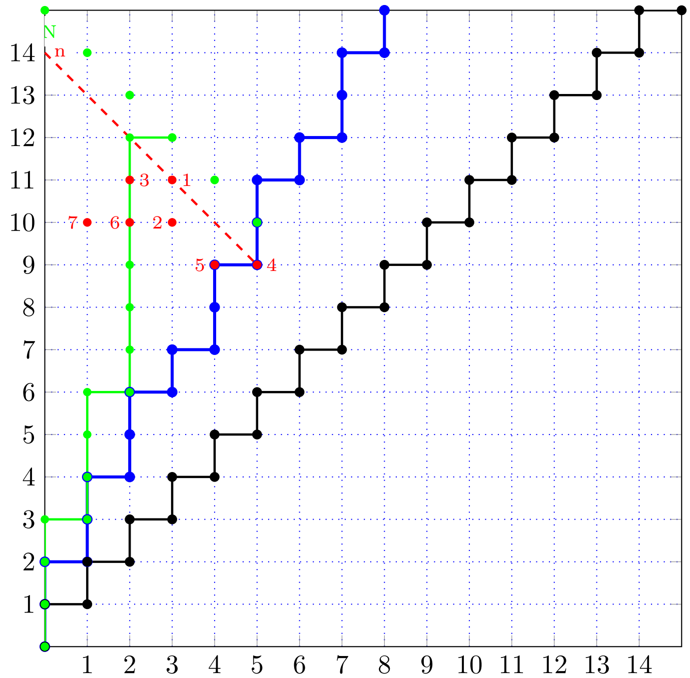

Figure 2: Diagram of transition paths relative to the Ceil boundary.

For $n = m + d > 2$, at the point (d, m) the probability that v_n is even is equal to $1/2$.

For the minimal solution v_0 of a transition list to be less than 2^{n-1}, and therefore equal to the value v_0 obtained for the restriction to the first $n-1$ transitions, it is necessary that the transition t_{n-1} be the "natural" transition taking v_{n-1} to v_n.

Conversely, for the minimal solution v_0 of a transition list to be greater than 2^{n-1}, and therefore not equal to the value v_0 obtained for the restriction to the first $n-1$ transitions, it suffices that the transition t_{n-1} is not the natural transition from v_{n-1} to v_n.

- For most points (interior points), such as point 1 with coordinates $(3, 11)$: the lists passing through point 1 originate either from point 2 or from point 3.
- For points on the boundary that are preceded by a single "East" step, such as point 4 with coordinates $(5, 9)$: the lists passing through point 4 originate only from point 5.
- In the case where the maximal value of m is taken to be $\lceil kN \rceil$, then for points with maximal ordinate, such as point 6 with coordinates $(2, 10)$: the lists passing through point 6 originate only from point 7.

In all these situations, a very large number of lists pass through the point, which means that the probability of v_{n-1} being even is always $1/2$. For the minimal solution v_0 of a transition list to satisfy $v_0 < 2^{n-1}$, it is necessary that the considered transition be the natural one, i.e., that v_{n-1} is odd at point 2 or even at point 3. Hence, statistically, there are twice as few lists after accounting for transition t_{n-1} whose minimal solution is less than 2^{n-1} as there were with minimal solution less than 2^n before accounting for this transition.

Repeating the same reasoning for all the last transitions, we conclude that for these sets of non-random and non-independent lists, the same result holds as in the random case.

We may note that translating the Ceil boundary horizontally to the right by prefixing it with $2p$ type 0 transitions does not alter the previous argument.

Under these circumstances, the *Random List Theorem* can be applied without any special modification.

Remarque 4.4 (Heuristic Approach to Establishing the Existence of Solutions)**.** Using the Central Limit Theorem, we observe that in the case $e = 6$, which is equivalent to $N - n = \lceil f(N) \rceil - 6$ and hence $2^{\lceil f(N) \rceil}/2^{N-n} = 64$, we obtain:

$$R_n < 64 + 8z = 96 < 128 = 2^7 \quad \text{for } z = 4.$$

This indicates that the number of minimal values is almost halved at each step when analyzing the last $N - n$ transitions. What initially appeared chaotic at the individual level becomes a smooth continuum when considering the system globally.

Even though there is no rigorous mathematical justification for it, the process being deterministic allows us to reasonably conjecture that, by adding 7+6=13 more steps (to account for the remaining fluctuations), we reach $R_{n-13} = 0$, meaning that there are no solutions $v_0 < 2^{n-13}$.

From this, we heuristically infer the following rule:

$$\boxed{\text{If } e < -7, \text{ then } R_n = 0 \text{ (no solution } v_0 < 2^n) \text{ with very high probability.}}$$

The probability is increasing as $e \ll -7$.

This rule is not mathematically rigorous, but it provides a useful intuition before applying formal reasoning with larger residual solutions.

Remarque 4.5. The validity of this estimate relies on the assumption that the sample of transition lists is drawn uniformly at random. Biases in the selection—such as favoring lists associated with small v_0—can significantly distort the statistical outcome. This has been confirmed by discrepancies observed in numerical experiments based on non-uniform or partitioned samplings.

Remarque 4.6. In earlier versions of this document (up to version 4.2.1 inclusive), the probabilistic reasoning relied on Corollary 3.8, which states that $\mathbb{P}(v_0 < 2^{N-k}) = \frac{1}{2^k}$. To bound the number of values $v_0 < 2^{N-k}$, denoted by R_k, the last k transitions were considered, and the Central Limit Theorem was used to estimate the associated binomial distribution at each step.

At each stage, R_k was bounded above and below around the expected proportion, using an interval centered at $n/2$ with growing width. This allowed a valid interval to be maintained at each step, but without control over the global error probability.

The weakness of this approach lies in the fact that extreme cases (beyond a certain number of standard deviations) were not taken into account. The assumption that R_k could not fall outside this interval relied on the idea that extreme cases could not occur, due to the underlying process being deterministic rather than purely random — a mathematically incorrect reasoning.

Indeed, if one fixes a threshold $z_k = 4$, corresponding to a local error $\varepsilon_k \approx 3.35 \times 10^{-4}$, then the probability that at least one of the k steps falls outside the interval is bounded above by $k\varepsilon_k$ (since the probability of a union is less than the sum of individual probabilities). For significant values of k (as used in the proof with $\alpha = 20$, $c_\alpha = 285$, $p = 100$, giving $k = c_\alpha \cdot \alpha - (c_\alpha + p) = 285 \times 20 - (285 + 100) = 5580$), this leads to a global error greater than 1, rendering the argument invalid.

In the current version, this mistake is addressed by consolidating the k steps into a single argument, relying on the fact that $R_k \sim \text{Bin}\left(nb, \frac{1}{2^k}\right)$ (see Theorem 3.1).

Remarque 4.7 (Comparison between the asymptotic (Central Limit Theorem) and rigorous (Berry–Esseen) approaches)**.** In informal reasoning, it is common to apply the Central Limit Theorem (CLT) to approximate a binomial distribution by a normal distribution as soon as the condition

$$nb \cdot p \gtrsim 30$$

is met. In our context, this allows filtering up to $N - n = f - 6$ when $nb = 2^f$, leaving only

$$R_{f-6} \approx 2^6 = 64$$

residual elements to analyze.

However, this approximation relies on asymptotic convergence without any explicit error bound. It is therefore not directly usable in a formal proof system such as Coq or Lean.

In contrast, the Berry–Esseen inequality provides a fully explicit bound on the deviation from the normal distribution. When applied with $\varepsilon = 10^{-3}$, it restricts the filtering depth to

$$N - n = f - 20,$$

leaving a much larger number of residual elements:

$$R_{f-20} \approx 2^{20} \approx 10^6.$$

This loss of efficiency is the price to pay for obtaining a **rigorous and formally justifiable** upper bound on the error probability, which is essential for formal verification.

Summary: the CLT provides sharper bounds but is not formally provable; Berry–Esseen is more conservative but suitable for rigorous proofs.

5 The Approximate Reduced Syracuse Sequence: (v'_n)

We consider an approximate version of the reduced Syracuse sequence, where the term $3v_n + 1$ is replaced with $3v_n$. This approximation is intuitively justified when v_0 is sufficiently large and n remains moderate, in which case the additive term $+1$ becomes negligible compared to the dominant multiplication by 3.

We construct a sequence (v'_n) that reproduces the same transition types (even or odd) as the exact sequence $(v_n)_{n\geq 0}$. It is defined by:

$$
\begin{cases}
v'_0 = v_0 > 0, \\
v'_{n+1} = \dfrac{v'_n}{2} & \text{if } v_n \text{ is even (type 0 transition),} \\
v'_{n+1} = \dfrac{3v'_n}{2} & \text{if } v_n \text{ is odd (type 1 transition).}
\end{cases}
$$

Note that the elements of the approximate sequence (v'_n) are generally not integers.

5.1 Decomposition of v_n in Terms of v'_n and a Rational Residue

Proposition 5.1. *For all $n \geq 0$, there exists a rational number $r_n \in \mathbb{Q}$ such that*

$$v_n = v'_n + r_n.$$

Proof. The proposition holds at $n = 0$ with $r_0 = 0$.

Assume it holds for some $n \geq 0$: $v_n = v'_n + r_n$. We prove it holds at $n+1$:

- **If v_n is even:**

$$v_{n+1} = \frac{v_n}{2} = \frac{v'_n + r_n}{2} = \frac{v'_n}{2} + \frac{r_n}{2} = v'_{n+1} + \frac{r_n}{2}.$$

  So $r_{n+1} = \frac{r_n}{2}$.

- **If v_n is odd:**

$$v_{n+1} = \frac{3v_n + 1}{2} = \frac{3(v'_n + r_n) + 1}{2} = \frac{3v'_n}{2} + \frac{3r_n + 1}{2} = v'_{n+1} + \frac{3r_n + 1}{2}.$$

  So $r_{n+1} = \frac{3r_n+1}{2}$.

By induction, the proposition holds for all $n \geq 0$. $\square$

Remarque 5.2. The sequence (r_n) can be defined recursively based on the transition types of (v_n):

$$
\begin{cases}
r_0 = 0 \\
r_{n+1} = \dfrac{r_n}{2} & \text{if } v_n \text{ is even} \\
r_{n+1} = \dfrac{3r_n + 1}{2} & \text{if } v_n \text{ is odd}
\end{cases}
$$

Note that the recurrence relation for (r_n) depends only on the parity pattern of (v_n) (i.e., the transition list), and not on the actual values of (v'_n) or the initial value v_0. It acts as a rational "residue" that encodes the discrepancy and allows reconstruction of the exact sequence (v_n) from its approximation (v'_n).

In particular, $r_n \geq 0$ for all $n \geq 0$.

Remarque 5.3. The sequence (r_n) remains small compared to (v'_n) when v_0 is large and n is moderate, justifying the approximation $v_n \approx v'_n$. This observation will be quantified in the next section to control the error term in applications of the approximate model.

5.2 Explicit Expression of v_n in Terms of v_0 and r_n

Proposition 5.4. *Let $\mathcal{L}(N, m, d) = (t_0, \ldots, t_{N-1})$ be a transition list of length N, and let m_n denote the number of type 1 transitions among its first n entries. Then for all $0 \leq n \leq N$, we have:*

$$v_n = \frac{3^{m_n}}{2^n} v_0 + r_n,$$

where (r_n) is the sequence defined in Proposition 5.1.

Proof. We recall that v'_n evolves under multiplicative factors of $1/2$ and $3/2$, depending on the transitions. After m_n type 1 transitions and $(n - m_n)$ type 0 transitions, we have:

$$v'_n = \left(\frac{3}{2}\right)^{m_n} \left(\frac{1}{2}\right)^{n-m_n} v_0 = \frac{3^{m_n}}{2^n} v_0.$$

Using $v_n = v'_n + r_n$, the result follows. $\square$

Remarque 5.5. This decomposition highlights a multiplicative factor $3^m/2^N$ depending only on the global structure of the transition list $\mathcal{L}(N, m, d)$, and a residue r_N depending solely on the positions of the type 1 transitions—not on the initial value v_0.

This is a key step toward applying the *Random List Theorem* discussed in Section 4.

5.3 Closed-Form Expression for r_n Based on Transitions

Théorème 5.6. *Let $\mathcal{L}(N, m, d)$ be a transition list of length N. For any $0 \leq n \leq N$, let m_n denote the number of type 1 transitions among the first n elements, and let $\text{ind}(i)$ denote the index (starting from 0) of the i^{th} type 1 transition in the list. Then:*

- *If $m_n = 0$, then $r_n = 0$.*
- *If $m_n > 0$, then:*

$$r_n = \frac{3^{m_n}}{2^n} \sum_{i=1}^{m_n} \frac{2^{\text{ind}(i)}}{3^i}.$$

Proof. We proceed by induction on n.

Base case: $n = 1$

- For $\mathcal{L} = (0)$, $m_1 = 0$ and $r_1 = 0$, so the formula holds (empty sum).
- For $\mathcal{L} = (1)$, $m_1 = 1$, $\text{ind}(1) = 0$:
$$r_1 = \frac{3^1}{2^1} \cdot \frac{2^0}{3^1} = \frac{1}{2}.$$
which matches the closed-form expression for r_1.

Induction step: Assume the formula holds at rank n. We show it holds at $n + 1$:

- If $t_n = 0$, then $m_{n+1} = m_n$, and:
$$r_{n+1} = \frac{r_n}{2} = \frac{3^{m_n}}{2^{n+1}} \sum_{i=1}^{m_n} \frac{2^{\text{ind}(i)}}{3^i}.$$
- If $t_n = 1$, then $m_{n+1} = m_n + 1$, and:
$$r_{n+1} = \frac{3r_n + 1}{2}.$$
Substituting r_n:
$$\begin{aligned} r_{n+1} &= \frac{1}{2^{n+1}} \left(3 \cdot \sum_{i=1}^{m_n} 3^{m_n - i} \cdot 2^{\text{ind}(i)} + 2^n \right), \\ &= \frac{1}{2^{n+1}} \left(\sum_{i=1}^{m_n} 3^{m_{n+1} - i} \cdot 2^{\text{ind}(i)} + 3^0 \cdot 2^n \right), \\ &= \frac{1}{2^{n+1}} \sum_{i=1}^{m_{n+1}} 3^{m_{n+1} - i} \cdot 2^{\text{ind}(i)}. \end{aligned}$$

Thus, the formula holds at $n + 1$. □

5.4 Effect of the Order of Type 0 Transitions on the Growth of r_n

Proposition 5.7. *Among all transition lists $\mathcal{L}(N, m, d)$ with m type 1 and d type 0 transitions, the final residue r_N satisfies:*

- *r_N is minimal when all type 1 transitions occur first (denoted* LRmin*),*
- *r_N is maximal when all type 0 transitions occur first (denoted* LRmax*).*

In particular:

$$r_N^{\min} = \frac{3^m}{2^N} - \frac{1}{2^d}, \qquad r_N^{\max} = \left(\frac{3}{2}\right)^m - 1.$$

Proof. From Theorem 5.6, we write:

$$r_N = \frac{3^m}{2^N} \sum_{i=1}^{m} \frac{2^{\text{ind}(i)}}{3^i}.$$

Shifting a type 0 transition earlier increases some indices $\text{ind}(i)$ without decreasing any. Since $x \mapsto 2^x$ is strictly increasing, r_N increases accordingly.

Minimum: all type 1 transitions first:

$$\text{ind}(i) = i - 1, \quad \text{for } 1 \leq i \leq m.$$

$$\begin{aligned} r_N^{\min} &= \frac{3^m}{2^N} \sum_{i=1}^{m} \frac{2^{i-1}}{3^i} = \frac{3^m}{2^N} \cdot \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} \cdot \frac{1}{3} = \frac{3^m}{2^N} \cdot \left(\frac{1 - \left(\frac{2}{3}\right)^m}{1 - \frac{2}{3}} \right) \cdot \frac{1}{3}. \\ &= \frac{3^m}{2^N} - \frac{1}{2^d}. \end{aligned}$$

Maximum: all type 0 transitions first:

$$\mathrm{ind}(i) = d + i - 1.$$

We factor out the 2^d term:

$$r_N^{\max} = \frac{3^m}{2^N} \sum_{i=1}^{m} \frac{2^{d+i-1}}{3^i} = \frac{2^d \cdot 3^m}{2^N} \sum_{i=1}^{m} \frac{2^{i-1}}{3^i}.$$

This sum is the same geometric series as above, hence:

$$r_N^{\max} = \frac{2^d \cdot 3^m}{2^N} \left(1 - \left(\frac{2}{3}\right)^m\right) = \left(\frac{3}{2}\right)^m - 1.$$

□

Remarque 5.8. The order of type 0 transitions can exponentially influence the residue r_N. Between the two extreme configurations:

$$\frac{r_N^{\max}}{r_N^{\min}} \approx 2^d.$$

This justifies focusing on subsets of transition lists where the residue r_N remains uniformly bounded. Such control is essential when comparing the exact trajectory (v_n) to its approximation (v'_n).

5.5 Final Residue for a Concatenation of Transition Lists

In this section, we study how the final residue $R_0 = r_{N_0}$ evolves when the transition list $\mathcal{L}_0 = \mathcal{L}(N_0, m_0, d_0)$ is obtained by concatenating a collection of sublists $\mathcal{L}_1, \ldots, \mathcal{L}_n$.

For each $k = 1, \ldots, n$, we define:

- $\mathcal{L}_k = \mathcal{L}(N_k, m_k, d_k)$: a transition list of length $N_k = m_k + d_k$,
- $F_k = \dfrac{3^{m_k}}{2^{N_k}}$: the multiplicative factor associated with $\mathcal{L}_k$,
- $R_k = r_{N_k}$: the rational residue associated with $\mathcal{L}_k$,
- $f_k = \prod_{i=1}^{k} \dfrac{1}{F_i}$: the reciprocal product of the F_i up to index k.

We recall from Proposition 5.4 that the final value of a block of transitions satisfies:

$$v_{(N_k, \mathcal{L}_k)} = F_k \cdot v_{(0, \mathcal{L}_k)} + R_k.$$

Proposition 5.9 (Concatenation formula for residues)**.** *Let $\mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2 + \cdots + \mathcal{L}_n$ be the successive concatenation of the lists $\mathcal{L}_k$. Then the final residue R_0 associated with $\mathcal{L}_0$ satisfies:*

$$R_0 = F_0 \cdot \sum_{k=1}^{n} f_k R_k, \quad \textit{where } F_0 = \prod_{k=1}^{n} F_k = \frac{3^{m_0}}{2^{N_0}}.$$

Proof. We prove the result by induction on the number n of concatenated blocks.

Base case: $n = 2$ Let $\mathcal{L}_0 = \mathcal{L}_1 + \mathcal{L}_2$. From Proposition 5.4, we have:

$$v_{(N_2, \mathcal{L}_2)} = F_2 \cdot v_{(0, \mathcal{L}_2)} + R_2, \quad \text{and} \quad v_{(0, \mathcal{L}_2)} = v_{(N_1, \mathcal{L}_1)} = F_1 \cdot v_{(0, \mathcal{L}_1)} + R_1.$$

Therefore:

$$\begin{aligned} v_{(N_0, \mathcal{L}_0)} &= F_2(F_1 v_0 + R_1) + R_2 \\ &= F_0 v_0 + F_2 R_1 + R_2, \\ \text{so} \quad R_0 &= F_2 R_1 + R_2 = F_0(f_1 R_1 + f_2 R_2). \end{aligned}$$

The general case follows by iterating this recurrence. □

6 Study of the Transition List $\mathrm{Ceil}(N)$

We focus here on a particular transition list, denoted $\mathrm{Ceil}(N)$, defined by a strict control on the proportion of type 1 transitions.

Définition 6.1. Let m_n be the number of type 1 transitions among the first n transitions in a list $\mathcal{L}(N, m, d)$. The list $\mathrm{Ceil}(N)$ is defined by the condition:

$$m_n = \left\lceil \frac{\ln 2}{\ln 3} \cdot n \right\rceil \quad \text{for all } 0 < n \leq N.$$

6.1 Threshold of the Trajectory: $v_n > v_0$ for All $0 < n \leq N$

Proposition 6.2. *Let (v_n) be the Syracuse sequence associated with the list* $\mathrm{Ceil}(N)$*. Then:*

$$v_n > v_0 \quad \text{for all } 0 < n \leq N.$$

Proof. By definition of $\mathrm{Ceil}(N)$, we have for all $n > 0$:

$$m_n = \left\lceil \frac{\ln 2}{\ln 3} \cdot n \right\rceil > \frac{\ln 2}{\ln 3} \cdot n.$$

This implies:

$$m_n \cdot \ln 3 > n \cdot \ln 2 \quad \text{if and only if} \quad 3^{m_n} > 2^n.$$

Using Proposition 5.4, we write:

$$v_n = \frac{3^{m_n}}{2^n} \cdot v_0 + r_n,$$

with $r_n \geq 0$. Therefore:

$$v_n > v_0,$$

as required. □

6.2 Bounding r_n in the $\mathrm{Ceil}(N)$ List

Théorème 6.3. *For the transition list* $\mathrm{Ceil}(N)$*, we have:*

$$\frac{m_n}{5} < r_n < m_n, \quad \text{for all } 0 < n < N \leq 10^6.$$

Proof. Direct numerical verification for $N \leq 10^6$

The entries in the tables below are sorted to highlight:

- Table 1 ($b_n = r_n - m_n$): most negative additive gaps;
- Table 2 ($a_n = r_n/m_n$): smallest multiplicative coefficients;
- Table 3 ($c_n = r_n/m_n$): largest multiplicative coefficients.

Table 1: b_n		Table 2: a_n		Table 3: c_n	
b_n	n	a_n	n	c_n	n
-0.5	1	0.2404498	780239	0.7213476	301994
-0.75	2	0.2404499	478245	0.7213469	603988
-1.344	5	0.2404504	176251	0.7213466	905982
-1.375	3	0.2404506	956490	0.7213466	125743
-1.562	4	0.2404506	654496	0.7213444	427737
-1.762	8	0.2404508	352502	0.7213442	75235
-2.508	7	0.2404516	830747	0.7213440	729731
-2.672	6	0.2404517	528753	0.7213435	251486
-3.321	10	0.2404522	226759	0.7213428	24727
-3.611	13	0.2404524	705004	0.7213421	553480
-3.688	16	0.2404526	403010	0.7213416	855474

For $N = 1000000$:

$$r_{1000000} = 198875.6767 \approx 2^{17.6} \approx 0.315 \cdot m_{1000000},$$

These results confirm that:

$$0.24 \cdot m_n < r_n < 0.72 \cdot m_n \quad \text{so} \quad \frac{m_n}{5} < r_n < m_n.$$

□

7 Answers to Questions and Conjectures Involving $\mathrm{sft}(n)$

7.1 Problem Analysis

The answers in this section concern the following points from the document "SE":

- **Title**: *"Is the Syracuse Falling Time Bounded by 12?"*
- **Conjecture 5.2**: *"We have* $\mathrm{sft}(n) \le 2$ *for all odd* $n \ge 2^{5000}$*."*
- **Question §3.2**: *"Again, we may ask whether* $\mathrm{sft}_{12}(n) = 1$ *holds for all odd* $n \ge 3$*."*

All these points refer to the function $\mathrm{sft}(n)$.

The complexity of the problem stems from the fact that the value $\mathrm{sft}(n)$ depends on the *jump* function $\mathrm{sjp}(n)$, which itself is based on the extracted sequence $\mathrm{syr}(n)$, containing only the odd values from the orbit of n under the application T. In general, it is therefore impossible to explicitly revert to T to compute $\mathrm{sjp}(n)$.

However, it is worth noting that $\mathrm{sjp}(n)$ is well-defined: if the sequence diverges starting from n, it contains infinitely many odd values, and thus at least ℓ of them. If it converges to a cycle, the cycle must contain at least one type-1 transition, and therefore also at least ℓ odd values.

We are interested here in the transition list $L(N, m, d)$ (see 2.2), which describes the sequence of parity bits or transition types associated with the orbit of n. The value $\mathrm{sjp}(n)$ corresponds to the ℓ-th odd value in this orbit, but the index of this value in the sequence $\{T^{(k)}(n)\}_{k\ge 0}$ is not constant.

In other words, the trajectory of n is a list $L(N, m, d)$, where $m = \ell$, $N = m + d$, and $T^{(N)}(n) = \mathrm{sjp}(n)$. Since d is not fixed, $\mathrm{sjp}(n)$ can be either much larger or much smaller than n.

- If d is large, then $\mathrm{sjp}(n) = T^{(N)}(n)$ is significantly smaller than n, and therefore may have fewer bits.
- If d is small, then $\mathrm{sjp}(n)$ is larger than n and may have more bits.

This variability makes the application of the *Random List Theorem* (see 4.2) challenging, since it is based on transition lists of fixed length N.

We therefore restrict our study to the integers n for which all values $\mathrm{sjp}^{(x)}(n)$, for $0 < x \le 12$, have the same number of bits as n and satisfy $\mathrm{sjp}^{(x)}(n) > n$. This restriction allows us to enumerate a subset of possible trajectories and to deduce a lower bound on the constant C sought in Conjecture 3.7.

We simply hope that this restriction is not too severe to prevent proving that $C \ge 13$.

The precise approach is divided into three main steps:

1. We determine the values of N_1 (trajectory lengths), and nb_1, the number of transition lists $L(N_1, m_1, d_1)$ (or the number of values of $n < 2^{N_1}$) such that $n \le T^{(N_1)}(n) = v_{N_1} \le 2n < 2^{N_1}$.
2. We then compute the number $nb = 2^{f(xN_1)}$ of transition lists $L(N = xN_1, m = xm_1, d = xd_1)$ (or the number of values $n < 2^N$), such that for all $0 < i \le x$, we have $T^{(iN_1)}(n) = v_{iN_1} > n$, and each such value has the same number of bits as n.
3. Finally, we apply the *Random List Theorem* (see 4.2), which allows us to conclude that there exist values of n such that $\mathrm{sft}(n) \ge x + 1$. We use Remark 4.3 to apply the theorem to this non-random set of transition lists.

   More precisely, if n is the minimal solution of a trajectory of length $N = xN_1$ such that for all $0 < i \le x$, we have $T^{(iN_1)}(n) > n$ and each $T^{(iN_1)}(n)$ has the same number of bits as n, then $T^{(N)}(n)$ also has the same number of bits as n.

   If there exist solutions n in the interval $[2^{\ell-1}, 2^{\ell})$ with $\ell = m_1$, then $T^{(N)}(n) = \mathrm{sjp}^{(x)}(n)$, since the correct number of odd transitions is present in each sublist of length N_1.

   By definition, $\mathrm{sft}(n)$ is the smallest value k such that $\mathrm{sjp}^{(k)}(n) < n$, which implies $\mathrm{sft}(n) \ge x + 1$.

   In other words, the existence of such solutions n in the interval $[2^{\ell-1}, 2^{\ell})$ with $\ell = m_1$ and $x = 12$ guarantees the existence of counterexamples to Conjecture 3.7 and provides a lower bound on the constant C.

   The *Random List Theorem* thus provides a perfectly suited tool for addressing this type of question.

7.2 Step 1: Determination of N_1 and nb_1

We determine the values of N_1, the trajectory lengths, and nb_1, the number of transition lists $L(N_1, m_1, d_1)$ (or the number of values of $n < 2^{N_1}$) such that $n \le T^{(N_1)}(n) = v_{N_1} \le 2n < 2^{N_1}$.

Possible Values of N_1

In this section, we use the approximate reduced sequence v', defined in Section 5, to model the iteration of the function T. Let m denote the number of type-1 transitions (that is, odd values) in the first p steps of the trajectory $\mathcal{O}_T(n)$. The following formula holds:

$$T^{(p)}(n) = \left(\frac{3^m}{2^p}\right) n + R_p,$$

where the residual term R_p is defined recursively by

$$R_0 = 0, \quad R_{p+1} = \begin{cases} R_p/2 & \text{if } T^{(p)}(n) \text{ is even,} \\ (3R_p + 1)/2 & \text{if } T^{(p)}(n) \text{ is odd.} \end{cases}$$

The term R_p does not explicitly depend on n. It depends only on the sequence of transitions, which allows it to be computed independently of the starting point, for each parity list.

Since $R_p \geq 0$ for all p, the following lower bound is immediate:

$$T^{(p)}(n) \geq \left(\frac{3^m}{2^p}\right) n.$$

We are thus interested in finding pairs (m, d), where $d = p - m$, such that the ratio $\frac{3^m}{2^{m+d}}$ exceeds 1. This condition guarantees that $T^{(p)}(n) > n$. At the same time, we seek ratios that remain close to 1, in order to control the growth.

Consider the inequality

$$\frac{3^m}{2^{m+d}} > 1.$$

This inequality holds if and only if

$$m \ln 3 - (m + d) \ln 2 > 0.$$

Since $\ln 3 - \ln 2 > 0$, dividing both sides by $d(\ln 3 - \ln 2)$ preserves the inequality, which gives

$$\frac{m \ln 3 - m \ln 2 - d \ln 2}{d(\ln 3 - \ln 2)} > 0.$$

Simplifying the numerator leads to

$$\frac{m}{d} > \frac{\ln 2}{\ln 3 - \ln 2}.$$

We set

$$X = \frac{\ln 2}{\ln 3 - \ln 2}.$$

The condition is therefore equivalent to

$$\frac{m}{d} > X.$$

We now consider upper approximations of the constant X by rational fractions m/d such that $\frac{3^m}{2^{m+d}} > 1$, while keeping the quotient as close to 1 as possible. These approximations are computed using the Stern-Brocot tree method. The following table presents the first such approximations, obtained from the supplementary test file [2]:

m	d	$N = m + d$	$3^m/2^N$
2	1	3	1.125
7	4	11	1.068
12	7	19	1.014
53	31	84	1.00209
359	210	569	1.0010664
665	389	1054	1.0000436
16266	9515	25781	1.0000255
31867	18641	50508	1.00000726

Table 2: Upper Approximations of X by m/d

These approximations provide candidate pairs (m_1, d_1), and trajectory lengths $N_1 = m_1 + d_1$, that are relevant for our study.

Since numerical tests have already covered the domain $\ell \leq 40$, particular attention will be given to the following values for N_1: 84, 569, 1054, 25781, and 50508. Additional pairs can be generated using the same method.

Condition $T^{(N_1)}(n) \leq 2n < 2^{N_1}$

For the condition $T^{(N_1)}(n) < 2^{N_1}$, it suffices to restrict the analysis to the minimal solutions of the transition lists of length N_1, in accordance with Lemma 3.4.

The condition $T^{(N_1)}(n) \leq 2n$ is more delicate. In particular, it is not possible to allow large oscillations of $T^{(i)}(n)$ around n for $0 < i < N_1$. Indeed, the residual term R_{N_1} is maximal when the list begins with d_1 type-0 transitions followed by m_1 type-1 transitions (the list $LRmax$), because leading type-0 transitions amplify R_p (see § 5.4).

In this case, we have the approximation:

$$T^{(N_1)}(n) \approx \frac{3^{m_1}}{2^{N_1}} \left(n + 2^{d_1}\right),$$

which imposes, in order to obtain $T^{(N_1)}(n) \leq 2n$, the condition $n < 2^{d_1}$. This bound is too restrictive in our context since $d_1 < m_1$.

According to Section 6.2, for the transition list $\mathrm{Ceil}(N_1)$, we have the bound

$$R_{N_1} < m_1,$$

and numerical computations provide the estimate

$$R_{1{,}000{,}000} < 2^{18}.$$

Therefore, in this context, the condition

$$T^{(N_1)}(n) \leq 2n$$

holds for the minimal solution n associated with $\mathrm{Ceil}(N_1)$.

It is useful to observe for later comparisons that

$$1{,}000{,}000 > 20 \times 25{,}781.$$

We will now restrict our attention to the set $\mathrm{E}(N_1)$, which consists of the transition lists $L(N_1, m_1, d_1)$ that satisfy

$$\mathrm{Ceil}(N_1) \preccurlyeq L(N_1, m_1, d_1),$$

where the parameter m_1 is defined by

$$m_1 = \left\lceil N_1 \cdot \frac{\ln 2}{\ln 3} \right\rceil.$$

Here, the symbol $\preccurlyeq$ refers to the partial order defined in Section 2.3.

Since $\mathrm{Ceil}(N_1)$ is the minimal element of $\mathrm{E}(N_1)$ with respect to this partial order, and since all the lists in $\mathrm{E}(N_1)$ contain the same number of type-1 transitions, it follows from Section 5.4 that $\mathrm{Ceil}(N_1)$ maximizes R_{N_1}.

Consequently, for every transition list in $\mathrm{E}(N_1)$, the inequality

$$T^{(N_1)}(n) \leq 2n$$

is satisfied when n is the minimal solution associated with the corresponding list.

Enumeration of the Admissible Transition Lists nb_1

The number of elements in the set $\mathrm{E}(N_1)$ is exactly nb_1.

For the transition list $\mathrm{Ceil}(N_1)$, the following property holds: for all integers p such that $0 < p \leq N_1$, we have

$$T^{(p)}(n) > n,$$

as stated in Proposition 6.2.

Since $\mathrm{Ceil}(N_1)$ is the minimal element of $\mathrm{E}(N_1)$ with respect to the partial order defined in Section 2.3, this property is also satisfied by all transition lists in $\mathrm{E}(N_1)$.

Without any constraint, the total number of possible transition lists of length N_1 containing m_1 type-1 transitions (and therefore $d_1 = N_1 - m_1$ type-0 transitions) would be $\binom{N_1}{m_1}$.

However, to account for the minimality condition imposed on $n = v_0$, we retain only one transition list among its N_1 circular permutations. This leads to:

$$nb_1 = \frac{1}{N_1} \binom{N_1}{m_1}, \quad \text{with } N_1 \in \{84, 569, 1054, 25781, \ldots\}.$$

Here are the first corresponding numerical values:

m_1	d_1	$N_1 = m_1 + d_1$	$\binom{N_1}{m_1} \approx 2^{cN_1}$	$nb_1 \approx 2^{(c - \log_2 N_1/N_1)N_1}$
53	31	84	$2^{0.90855N_1}$	$2^{0.83245N_1}$
359	210	569	$2^{0.94143N_1}$	$2^{0.92536N_1}$
665	389	1054	$2^{0.94493N_1}$	$2^{0.93540N_1}$
16266	9515	25781	$2^{0.94966N_1}$	$2^{0.94909N_1}$
31867	18641	50508	$2^{0.94980N_1}$	$2^{0.94950N_1}$

Table 3: Values of nb_1

7.3 Step 2: Determination of the Number $nb = 2^{f(N)}$ of Transition Lists $L(N = xN_1, m = xm_1, d = xd_1)$

We compute the number $nb = 2^{f(xN_1)}$ of transition lists $L(N = xN_1, m = xm_1, d = xd_1)$ (or the number of values $n < 2^N$), such that for all $0 < i \le x$, we have $T^{(iN_1)}(n) = v_{iN_1} > n$, and each such value has the same number of bits as n.

According to Lemma 3.4, for each trajectory of length $N = xN_1$, the condition $n < 2^N$ is automatically satisfied as soon as n is a minimal solution.

We consider the trajectories $L(N, m, d)$ obtained by concatenating x transition lists L_k, each of length N_1. These lists may differ from one another, but each belongs to the set $\mathrm{E}(N_1)$, as defined in the previous step.

By construction, for any such trajectory L, we have:

$$T^{(iN_1)}(n) > n \quad \text{for all } 0 < i \le x,$$

and, more strongly,

$$T^{(j)}(n) > n \quad \text{for all } 0 < j \le N = xN_1,$$

which implies that the falling time $\sigma(n)$ satisfies $\sigma(n) > N$.

We must now verify that, for all $0 < i \le x$, the value $T^{(iN_1)}(n)$ retains the same number of bits as n.

To do so, we use the general formula from paragraph 5.5 giving the residual term R_N when L is the concatenation of x sublists L_k:

We have the following formula for the residual term:

$$R_N = \frac{3^m}{2^N} \sum_{k=1}^{x} f_k R_k,$$

where

$$f_k = \prod_{i=1}^{k} \frac{1}{F_i}, \quad \text{and} \quad F_i = \frac{3^{m_i}}{2^{N_i}}.$$

In our specific case, for all k, we have $m_k = m_1$ and $N_k = N_1$, so that $F_i = F_1$ for every i. Moreover, the inequality

$$R_k \le R_{\mathrm{Ceil}(N_1)} \le m_1$$

holds for all k, and we observe that

$$\frac{3^m}{2^N} = F_1^x.$$

We deduce the following bound for R_N:

$$R_N \le R_{\mathrm{Ceil}(N_1)} \sum_{k=1}^{x} F_1^{x-k} \le m_1 \sum_{k=0}^{x-1} F_1^k.$$

This yields the inequality

$$R_N \le m_1 \cdot \frac{1 - F_1^x}{1 - F_1}.$$

Since F_1 is approximately equal to 1, it is convenient to set

$$u_1 = 1 - F_1.$$

With this notation, we can approximate

$$F_1^x \approx 1 - xu_1.$$

Explicitly, this leads to the estimate:

$$R_N \le m_1 \cdot x \cdot F_1,$$

where

$$F_1 = \frac{3^{m_1}}{2^{N_1}} \approx 1, \quad \text{and} \quad m_1 = \left\lceil N_1 \cdot \frac{\ln 2}{\ln 3} \right\rceil.$$

As a consequence, we obtain the following inequality for the xN_1-th iterate of T:

$$n \leq T^{(xN_1)}(n) = F_1^x \cdot n + R_N \leq F_1^x \cdot n + m_1 \cdot x \cdot F_1.$$

This expression shows that if $T^{(xN_1)}(n)$ remains in the same power of two as n, then the intermediate terms $T^{(iN_1)}(n)$ do as well, since F_1^x and R_N grow moderately with x.

The following table provides typical values for $x = 20$:

m_1	d_1	$N_1 = m_1 + d_1$	${F_1}^x$	$m_1 \cdot x \cdot F_1$
53	31	84	1.04265	$1{,}062 \approx 2^10$
359	210	569	1.02155	$7{,}188 \approx 2^{12.81}$
665	389	1054	1.000873	$13{,}300 \approx 2^{13.7}$
16266	9515	25781	1.000509	$325{,}328 \approx 2^{18.31}$
31867	18641	50508	1.000145	$637{,}345 \approx 2^{19.28}$

Table 4: Values of ${F_1}^x$ and upper bound for R_N when $x = 20$

In all cases, we observe that $T^{(xN_1)}(n) < 1.05 \cdot n$ for $x = 20$, and much less for the larger values of N_1. Moreover, if we consider integers n chosen in the interval $[2^{m_1-1}, 2^{m_1})$, the residual term R_N is negligible in comparison, which ensures the stability of the number of bits.

Under the reasonable assumption of a fairly uniform distribution of minimal values n in the intervals $[2^i, 2^{i+1})$, we can estimate that in at least half of the cases, $T^{(xN_1)}(n)$ retains the same number of bits as n.

The total number of transition lists is therefore nb_1^x, and we retain approximately half of them, giving $nb = \frac{1}{2} \cdot nb_1^x$, which leads to:

m_1	d_1	$N_1 = m_1 + d_1$	$nb = 2^{f(xN_1)}$
53	31	84	$2^{0.83245 \cdot xN_1 - 1}$
359	210	569	$2^{0.92536 \cdot xN_1 - 1}$
665	389	1054	$2^{0.93540 \cdot xN_1 - 1}$
16266	9515	25781	$2^{0.94909 \cdot xN_1 - 1}$
31867	18641	50508	$2^{0.9495 \cdot xN_1 - 1}$

Table 5: Values of $nb = 2^{f(xN_1)}$ as a Function of x

7.4 Step 3: Application of the Random List Theorem to Lower Bound C

Suppose there exists a solution $n \in [2^{\ell-1}, 2^{\ell})$ satisfying the properties established in the previous steps, with $\ell = m_1$ and $N = xN_1$. In this case, for all $0 < i \leq x$, we have:

$$2^{\ell-1} \leq T^{(iN_1)}(n) < 2^{\ell},$$

and therefore, due to the exact number of odd transitions per sublist, each $T^{(iN_1)}(n)$ corresponds to $\mathrm{sjp}^{(i)}(n)$.

Since $T^{(xN_1)}(n) = \mathrm{sjp}^{(x)}(n) > n$, it follows that:

$$\mathrm{sft}(n) \geq x + 1.$$

The *Random List Theorem* (paragraph 4.2) provides a particularly effective tool for addressing this type of question: it allows us to estimate the existence of such solutions n in the interval $[0, 2^{m_1})$, with $m_1 = \lceil N_1 \cdot \ln 2 / \ln 3 \rceil$ and $N = xN_1$.

Approximately half of the integers in this interval also satisfy $n \in [2^{m_1-1}, 2^{m_1})$, which makes it possible to focus on the relevant cases.

In the diagram above:

- The blue line represents the JGL boundary (a constraint to be respected);
- The green line is a valid trajectory, always staying above JGL;
- The black line is the classical boundary of the Catalan triangle (without the JGL constraint);
- Green point mark the endpoint of valid trajectories;
- Blue points represent trajectories admissible in the Catalan triangle but invalid under the JGL constraint;
- Black points lie outside both domains.

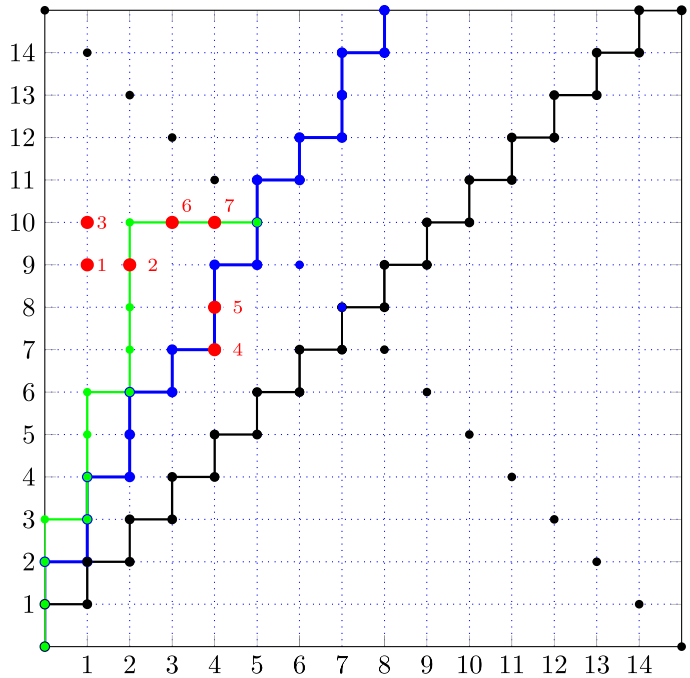

Figure 3: Diagram of transition paths in $\mathrm{Up}(N, v_0)$ relative to the JGL boundary.

Remarque 7.1 (Exhaustive Set of Lists)**.** Let us consider the set of transition lists $\mathcal{L}(n, m, d)$ of length n that lie above the JGL boundary. We distinguish three cases:

- For lists whose endpoint is represented by a point (such as point 1 in Figure 3) strictly inside the region bounded by the JGL list: all such lists can be extended by any additional transition and will therefore terminate at either point 2 or point 3. Consequently, regardless of the proportion of even values v_n at point 1, the probability that the extended list satisfies $v_0 < 2^n$ is exactly half that of the original list at point 1.
- For lists whose endpoint corresponds to a point (such as point 4) that has the same number of type 1 transitions as the JGL list and where the next transition in the JGL list is of type 1: this is an nontrivial case where, at point 5, the probability that $v_0 < 2^n$ is not exactly half that of point 4, but rather multiplied by the proportion of odd values v_n at point 1, which is approximately 1/2.

  Unfortunately, it is precisely near such points 4 (which occur with relative frequency around $1/n$ compared to points 1), close to the JGL boundary, that the number of lists is maximal, since the binomial coefficient $\binom{n}{m}$ decreases rapidly when $m > n/2$. In practice, however, discrepancies from exact independence tend to cancel out across successive transitions.
- For lists whose endpoint corresponds to a point (such as point 6) that already has the same final number of type 1 transitions as the JGL list: this is a nontrivial case where, at point 7, the probability that $v_0 < 2^n$ is not exactly half that of point 6, but rather multiplied by the proportion of even values v_n at point 1, which is approximately 1/2. Fortunately, the number of such lists is minimal. As in the previous case, practical discrepancies from exact independence tend to cancel out across successive transitions.

While we acknowledge that the lists are not formally independent, we still apply the *Random List Theorem*, as the observed discrepancies are so large that the resulting contradiction may be viewed as effectively formal.

Asymptotic Case. In the limit as $N_1 \to \infty$, we use the asymptotic approximation:

$$\binom{N_1}{N_1 \cdot \ln 2/\ln 3} \sim 2^{0.949956 N_1}.$$

We then seek to satisfy the inequality (applying the theorem with $n = m_1$ and $N = xN_1$):

$$e = m_1 - xN_1 + f(xN_1) \geq 33,$$

which corresponds to a minimum of 4,294,522,559 $> 2^{31}$ solutions.

This condition becomes:

$$N_1 \cdot \frac{\ln 2}{\ln 3} - xN_1 + 0.949956xN_1 - 1 \geq 12,$$

from which we deduce:

$$x \leq \frac{-13/N_1 + \ln 2/\ln 3}{1 - 0.949956} \approx 12.6075.$$

This proves the existence of solutions for $x = 12$, but not beyond, in this restricted framework.

Concrete Examples. We numerically evaluate the expression $e = m_1 - xN_1 + f(xN_1)$ with $x = 12$:

- For $m_1 = 53$, $N_1 = 84$:

$$53 - 12 \times 84 + 0.83245 \times 12 \times 84 - 1 = -116.89 < -7.$$

  Therefore, based on the rule *Heuristic Approach to Establishing the Existence of Solutions* (see Remark 4.4), it is highly unlikely that any solution exists.

- For $m_1 = 359$, $N_1 = 569$:

$$-151.64 < -7.$$

  Therefore, this case also admits no solution.

- For $m_1 = 665$, $N_1 = 1054$:

$$-153.06 < -7.$$

  Therefore, this case also admits no solution.

- For $m_1 = 16266$, $N_1 = 25781$:

$$16266 - 12 \times 25781 + 0.94909 \times 12 \times 25781 - 1 = 594.87 \gg 33.$$

  Therefore, by applying the *Random List Theorem* together with the Berry–Esseen inequality (see 4.2), we can assert that, with probability greater than $1 - 10^{-5}$, there are at least 4,294,522,559 $> 2^{31}$ solutions below 2^{m_1}. More precisely, we even have:

$$2^{31} \times 2^{m_1 - 12N_1 + f(12N_1) - 33} \approx 2^{592}.$$

  Since we target the interval $[2^{m_1-1}, 2^{m_1})$, there remain about 2^{591} useful solutions. Since the distribution is binomial, we are formally guaranteed that at least one solution exists.

For an infinite number of values of $N_1 > 25781$ (corresponding to upper approximations of X), the number of solutions continues to grow.

Remarque 7.2.

- The smallest solutions satisfying $\mathrm{sft}(n) \geq 13$ lie in the interval $[2^{16265}, 2^{16266})$. This is counterintuitive but is explained by the asymptotic growth of $\binom{N_1}{m_1}$.

- Even though there are about 2^{591} solutions for $\ell = 16266$, it is incorrect to believe that they are easier to find. The total number of trajectories to test is at least:

$$\binom{25781}{16266}^{12} \approx 2^{0.94966 \times 25781 \times 12} = 2^{293798}.$$

  Under these conditions, a direct test over all integers in $2^{16265} \leq n < 2^{16266}$ would be preferable, requiring only 2^{16265} evaluations. However, this remains infeasible today, since the probability of success for a random test in this interval is approximately:

$$\frac{2^{591}}{2^{16265}} = 2^{-15674} \approx 10^{-4718}.$$

  This estimation explains why empirical verifications, limited to $n < 2^{68}$, are necessarily insufficient to detect such counterexamples.

7.5 Conclusion on the Conjectures Considered

- **Conjecture 3.7**, cited in the title of the article as: *"There exists $C \geq 10$ such that $\mathrm{sft}(n) \leq C$ for all $n \equiv 3 \bmod 4$"*: If an upper bound C exists for the function $\mathrm{sft}(n)$, then necessarily $C \geq 13$. The authors were therefore correct to include a question mark in the title of their article, since the values $C = 11$ or 12 are insufficient.

- **Conjecture 5.2**, asserting that $\mathrm{sft}(n) \leq 2$ for all odd integers $n \geq 2^{5000}$, is contradicted. Indeed, there exist integers n satisfying $2^{5000} < n < 2^{16266}$ such that $\mathrm{sft}(n) \geq 13$. Moreover, the theoretical reasoning shows the existence of infinitely many such counterexamples.

- **Question §3.2**: For the integers n constructed in this analysis, the value $\mathrm{sft}_x(n)$ coincides with the x-th iterate of the application sjp in most cases. Consequently, there exist integers n such that $\mathrm{sjp}_{12}(n) > n$, which implies $\mathrm{sft}_{12}(n) > 1$. The question posed on page 13 of [3], *"Again, we may ask whether* $\mathrm{sft}_{12}(n) = 1$ *holds for all odd* $n \geq 3$*"*, therefore receives a negative answer.

These results rely on a purely theoretical approach, independent of any explicit search. The existence of counterexamples is guaranteed, although they are inaccessible to direct computation within the bounds explored so far ($n < 2^{68}$).

The approach presented is based on a restricted subset of trajectories. It is plausible that one could go beyond $C = 13$, but this would require a more general analysis of the relationships between $\mathrm{sjp}(n)$ and T.

Finally, the probabilities associated with the effective occurrence of such counterexamples within the classical test ranges are extremely small, which explains their absence from experimental checks.

8 Answers to Questions and Conjectures Involving $\mathrm{ft}(n)$

The questions addressed in this section correspond to the following points in the document *SE*:

- **Conjecture 5.1**: *"We have* $\mathrm{ft}(n) \leq 4$ *for all* $n \geq 2^{500}$*."*
- **Page 8**: *"We do not know whether* $\mathrm{ft}(n) \geq 17$ *is at all reachable."*
- **Page 9**: *"Is it true that* $\mathrm{ft}_{18}(n) = 1$ *for all* $n \geq 3$*?"*
- **Challenge**: *"If you do find any* $n \geq 2^{500}$ *satisfying* $\mathrm{ft}(n) \geq 5$*, please e-mail it to the authors."*

All these points concern the function $\mathrm{ft}(n)$, defined from the *jump* application $\mathrm{jp}(n)$, which retains only specific values from the orbit of n under the action of T. The structure of jp prevents any explicit expression of $\mathrm{ft}(n)$ directly in terms of the T-iteration.

As before, we focus on integers n whose successive iterates $\mathrm{jp}^{(i)}(n)$ retain the same number of bits ℓ, with $\mathrm{jp}^{(i)}(n) > n$ for all $i \leq x$.

The method is analogous to that used for $\mathrm{sft}(n)$, replacing N_1 with ℓ in the trajectory constructions. We again apply the *Random List Theorem* (see 4.2), this time for trajectories of length $N = x\ell$, and evaluate the existence of solutions $n < 2^\ell$ such that $\mathrm{ft}(n) \geq x + 1$.

Although we consider only a restricted subset of possible trajectories, we still obtain significant lower bounds for the value of $\mathrm{ft}(n)$.

Asymptotic Analysis. In the limit as $\ell \to \infty$, the number of admissible trajectories grows like $2^{0.949956x\ell}$. We therefore seek to solve the inequality:

$$e = \ell - x\ell + f(x\ell) \geq 12,$$

which guarantees at least $2^{10} = 1024$ solutions. This leads to:

$$\ell - x\ell + 0.949956 \cdot x\ell - 1 \geq 12, \quad \Rightarrow \quad x \leq \frac{-13/\ell + 1}{1 - 0.949956} \approx 19.98.$$

We can therefore assert that there exist integers n such that $\mathrm{ft}(n) \geq 20$.

Explicit Evaluations. Here are some concrete cases:

1. For $\ell = 569$ and $x = 4$:

$$\ell - x\ell + 0.92536 \times x\ell - 1 = 398.12 \gg 33 \quad \Rightarrow \text{at least } 2^{390} \text{ solutions.}$$

2. For $\ell = 25781$ and $x = 18$:

$$\ell - x\ell + 0.94909 \times x\ell - 1 = 2154.81 \gg 33 \quad \Rightarrow \text{more than } 2^{2150} \text{ solutions.}$$

Conclusions Related to the Statements in [3]

- **Conjecture 5.1** is contradicted starting from $\ell = 569 > 500$, since we theoretically observe integers n such that $\mathrm{ft}(n) \geq 5$.
- The answer to the **question on page 8** is positive: according to case 3), it is possible to reach (and exceed) $\mathrm{ft}(n) \geq 17$.
- The answer to the **question on page 9** is negative: for many values of n, we have $\mathrm{jp}^{(18)}(n) > n$, so $\mathrm{ft}_{18}(n) > 1$.

- Regarding the **challenge**, the results obtained imply the theoretical existence of infinitely many counterexamples. However, an exhaustive test over $n \in [2^{\ell-1}, 2^{\ell})$ for $\ell = 569$ remains infeasible. The probability that a random test in this interval identifies a solution is approximately:

$$\frac{2^{390}}{2^{569}} = 2^{-179} \approx 1.5 \times 10^{-54}.$$

  This explains why the authors will probably never receive the email they invite readers to send.

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[3] S. Eliahou, J. Fromentin, and R. Simonetto, "Is the Syracuse Falling Time Bounded by 12?" (2021). `https://doi.org/10.48550/arXiv.2107.11160`

[4] J. C. Everett, Iteration of the number-theoretic function $f(2n) = n$, $f(2n+1) = 3n+2$, *Adv. Math.* **25** (1) (1977), 42–45.

[5] R. Terras, A stopping time problem on the positive integers, *Acta Arith.* **30** (3) (1976), 241–252.